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Invited paper

Data distributions in magnetic resonance images: A review

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ABSTRACT

Many image processing methods applied to magnetic resonance (MR) images directly or indirectly rely on prior knowledge of the statistical data distribution that characterizes the MR data. Also, data distributions are key in many parameter estimation problems and strongly relate to the accuracy and precision with which parameters can be estimated. This review paper provides an overview of the various distributions that occur when dealing with MR data, considering both single-coil and multiple-coil acquisition systems. The paper also summarizes how knowledge of the MR data distributions can be used to construct optimal parameter estimators and answers the question as to what precision may be achieved ultimately from a particular MR image.

Introduction

Magnetic resonance imaging (MRI) is the diagnostic tool of choice in biomedicine. It is able to produce high-quality three-dimensional images containing an abundance of physiological, anatomical and functional information. A voxel's grey level within an MR image represents the amplitude of the radio frequency signal coming from the hydrogen nuclei (protons) within that voxel. To draw reliable diagnostic conclusions from MR images, visual inspection alone is often insufficient. Quantitative data analysis is required to extract the information needed. Such an analysis can almost without exception be formulated as a parameter estimation problem. The parameters of interest can simply be the values of the true MR signal underlying the noise corrupted data points [1–3], but also proton densities (in the construction of proton density maps [4,5]), relaxation time constants (in the construction of T_1, T_2 and T_2^* maps [4–11]) or diffusion parameters (in diffusion MRI) [12–14]. Different estimators can be constructed to estimate one and the same parameter, but it is well known that the best estimators (in terms of accuracy and precision) are constructed by properly taking the statistical distribution of the data into account. Hence, knowledge of the MRI data distribution is of vital importance.

This review paper gives an overview of the various distributions that occur when dealing with MR data, considering both single-coil and multiple-coil systems. The paper also summarizes how knowledge of these distributions can be used to construct optimal estimators and to answer the question as to what precision may be achieved ultimately from a particular MR image.

The organization of the paper is as follows. Section 2 briefly reviews MR signal detection and introduces a statistical model of the complex valued raw MR data acquired in the so-called *k*-space (i.e., the spatial frequency domain). Section 3 then describes the statistical distribution of the reconstructed images in the spatial domain, assuming the data have been acquired using a single-coil system. Complex images as well as magnitude and phase images, which can be constructed from the complex images straightforwardly, are considered. Since image acquisition with multiple coils is becoming more and more common nowadays, Section 4 describes the distribution of complex and magnitude images acquired with multiple-coil systems. Section 5 reviews the theory that explains how knowledge of the distribution of the MR images can be used to (i) derive a lower bound on the variance of any unbiased estimator of parameters from these images (the so-called Cramér-Rao Lower Bound), and (ii) to construct the maximum likelihood (ML) estimator, which attains this lower bound at least asymptotically. In Section 6, this theory is applied to (i) derive the CRLB for unbiased estimation of the underlying true signal amplitude from (single-coil) magnitude images and, (ii) derive the ML estimator for this estimation problem. In Section 7, conclusions are drawn.

Notation: throughout this paper, vectors will be underlined and matrices will be expressed in capital letters. Furthermore, random

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variables (RVs) will be expressed in bold face. The operators $\mathbb{E}[\cdot]$ and $Var(\cdot)$ denote the expectation and variance of a random variable, respectively. The real part of a complex valued variable z is denoted as z_R and the imaginary part as z_I. The complex conjugate of X is denoted as X^* and the transpose and complex conjugate transpose of X are denoted as X^T and X^H, respectively. Furthermore, we use $f_{\mathbf{x}}(x)$ to denote the probability density function (PDF) of the random variable $\mathbf{x}$. The conditional PDF of the RV $\mathbf{x}$ conditioned on the RV $\mathbf{y}$ is denoted as $f_{\mathbf{x}|\mathbf{y}}(x|y)$. The modified Bessel function of the first kind of order ν is denoted as $I_\nu(\cdot)$. The symbol $\imath$ denotes $\sqrt{-1}$.

Signal detection and modeling

This section briefly reviews the mathematics behind signal detection in MRI and describes the concepts of signal demodulation and quadrature detection. The section is to a large extent based on Refs. [15–19]. For a more comprehensive description, the reader is referred to those references. The final purpose of the section is to introduce a statistical model of the detected MR signal.

Modeling the noise free signal

In MRI, an object is placed in a strong static, external, homogenous magnetic field $\underline{B}_0$ that polarizes the protons in the object, yielding a net magnetic moment oriented parallel to $\underline{B}_0$. Let's assume that $\underline{B}_0$ points in the z-direction. Next, a radio frequency pulse is applied that generates another, oscillating magnetic field $\underline{B}_1$ perpendicular to $\underline{B}_0$. This so-called excitation field tips away the net magnetic moment from the z-axis, producing a magnetization component transverse to the static field. This transverse magnetization component precesses at the so-called Larmor frequency

$$\omega_0 = \gamma\left|\underline{B}_0\right|,$$

with γ the gyromagnetic ratio. This precessing magnetization vector induces a voltage in the receiver/detector coil (a conducting loop). Spatial information can be encoded in the received signal by augmenting $\underline{B}_0$ with additional, spatially varying magnetic fields. These so-called gradient fields vary linearly in space and are denoted as G_x, G_y and G_z. For example, when G_x is applied, the strength of the static magnetic field will vary with position in the x-direction as $\left|\underline{B}_z(x)\right| = \left|\underline{B}_0\right| + G_x x$, where the subscript z is used to denote that the magnetic field points in the z-direction. In this way, gradient fields can be used to make the precession frequency vary linearly in space. MRI signal detection is based on Faraday's law of electromagnetic induction and the principle of reciprocity [15]. Assuming a static inhomogeneous magnetic field pointing in the z-direction, the (noise free) voltage signal $v(t)$ in the receiver coil is related to the transverse magnetization distribution $\underline{M}_{xy}(\underline{r},t)$ of the object by the expression [15]

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t/T_2\left(\underline{r}\right)} \cos\left[-\omega\left(\underline{r}\right)t + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r} \tag{1}$$

with $\underline{r} = (x,y,z)^T$ the position in the laboratory frame, $t = 0_+$ the time instant immediately after the excitation pulse, $\omega(\underline{r})$ the free precession frequency, T_2 a relaxation time constant, $\underline{B}_{r,xy}(\underline{r})$ the detection sensitivity of the coil, $\phi_r(\underline{r})$ the reception phase angle, and $\phi_e(\underline{r})$ the initial phase shift introduced by RF excitation. The detection sensitivity $\underline{B}_{r,xy}(\underline{r})$ is defined as the xy vector component of the field generated at $\underline{r}$ by a unit current in the coil. The phase contributions $\phi_r(\underline{r})$ and $\phi_e(\underline{r})$ take a value between 0 and 2π depending on the direction of, respectively, $\underline{B}_{r,xy}(\underline{r})$ and $\underline{M}_{xy}(\underline{r},0_+)$ in the transverse plane [15]. Assuming that a frequency encoding gradient G_x was turned on during the signal read out (i.e., during data acquisition), we have

$$\omega\left(\underline{r}\right) = \omega_0 + \Delta\omega\left(\underline{r}\right), \tag{2}$$

with

$$\Delta\omega\left(\underline{r}\right) = \gamma G_x x, \tag{3}$$

where $\Delta\omega(\underline{r})$ is the spatially varying resonance frequency in the Larmor-rotating frame, i.e., the coordinate system whose transverse plane is rotating clockwise at an angular frequency ω_0 [15]. Furthermore, if we assume that a so-called phase encoding gradient G_y was turned on for a time interval T_{pe} before the signal read out, we have to add a position dependent initial phase contribution $\phi_{pe}(\underline{r})$ to $v(t)$:

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t/T_2\left(\underline{r}\right)} \cos\left[-\omega\left(\underline{r}\right)t - \phi_{pe}\left(\underline{r}\right) + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r}, \tag{4}$$

with

$$\phi_{pe}\left(\underline{r}\right) = \gamma G_y y T_{pe}. \tag{5}$$

MR image reconstruction concerns the inverse problem of reconstructing the transverse magnetization distribution $\underline{M}_{xy}(\underline{r},t)$ from the voltage signal $v(t)$. If we assume that a slice selective gradient G_z has been applied in the z-direction during the excitation period, only protons in the selected slice (at, say, $z = z_0$) are excited, so that $\underline{M}_{xy}(x,y,z_0,t) = \underline{M}_{xy}(x,y,t)$ [18]. The MRI reconstruction problem then reduces to producing a spatial map in two dimensions. Assuming that $\left|\underline{M}_{xy}(\underline{r},0_+)\right| e^{-t/T_2(\underline{r})}$ is relatively constant during data acquisition, Eq. (4) can be simplified to

$$v(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{acq}\right)\right| \cos\left[-\omega\left(\underline{r}\right)t - \phi_{pe}\left(\underline{r}\right) + \phi_e\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right] d\underline{r} \tag{6}$$

with t_{acq} the time at the center of the acquisition and

$$\underline{M}_{xy}\left(\underline{r},t_{acq}\right) = \left|\underline{M}_{xy}\left(\underline{r},0_+\right)\right| e^{-t_{acq}/T_2\left(\underline{r}\right)} e^{\imath\phi_e\left(\underline{r}\right)}. \tag{7}$$

In practice, $\Delta\omega(\underline{r}) \ll \omega_0$ and $v(t)$ is a *high frequency* bandpass signal centered about the frequency $\pm\omega_0$. The high-frequency nature of $v(t)$ may cause unnecessary problems for electronic circuits in later processing stages [15]. In practice, these problems are circumvented by exploiting the following property of the bandpass signal $v(t)$. It can be shown that the bandpass signal $v(t)$ can be represented as [19]

$$v(t) = \Re[\tilde{v}(t)\exp(\imath\omega_0 t)], \tag{8}$$

where $\Re[z]$ denotes the real part of the complex number z and $\tilde{v}(t)$ is the so-called complex envelope of $v(t)$, which can be written as

$$\tilde{v}(t) = \tilde{v}_R(t) + \imath\tilde{v}_I(t), \tag{9}$$

with

$$\tilde{v}_R(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right|\cos\left[-\Delta\omega\left(\underline{r}\right)t - \phi_{\text{pe}}\left(\underline{r}\right) + \phi_{\text{e}}\left(\underline{r}\right) - \phi_r\left(\underline{r}\right) + \frac{\pi}{2}\right]d\underline{r}, \tag{10}$$

and

$$\tilde{v}_I(t) = \int_{\text{object}} \omega\left(\underline{r}\right)\left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right|\sin\left[-\Delta\omega\left(\underline{r}\right)t - \phi_{\text{pe}}\left(\underline{r}\right) + \phi_{\text{e}}\left(\underline{r}\right) - \phi_{\text{r}}\left(\underline{r}\right) + \frac{\pi}{2}\right]d\underline{r}. \tag{11}$$

The signal $\tilde{v}_R(t)$ is called the *in-phase* component and $\tilde{v}_I(t)$ is called the *quadrature* component of $v(t)$. Note that it follows from Eqs. (8) and (9) that the original bandpass signal $v(t)$ can be written in terms of $\tilde{v}_R(t)$ and $\tilde{v}_I(t)$ as:

$$v(t) = \tilde{v}_R(t)\cos(\omega_0 t) - \tilde{v}_I(t)\sin(\omega_0 t). \tag{12}$$

Since in practice $\Delta\omega(\underline{r}) \ll \omega_0$, Eqs. (10) and (11) can be simplified to

$$\tilde{v}_R(t) = \omega_0 \int_{\text{object}} \left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right|\cos\left[-\Delta\omega\left(\underline{r}\right)t - \phi_{\text{pe}}\left(\underline{r}\right) + \phi_{\text{e}}\left(\underline{r}\right) - \phi_{\text{r}}\left(\underline{r}\right) + \frac{\pi}{2}\right]d\underline{r}, \tag{13}$$

and

$$\tilde{v}_I(t) = \omega_0 \int_{\text{object}} \left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right|\sin\left[-\Delta\omega\left(\underline{r}\right)t - \phi_{\text{pe}}\left(\underline{r}\right) + \phi_{\text{e}}\left(\underline{r}\right) - \phi_{\text{r}}\left(\underline{r}\right) + \frac{\pi}{2}\right]d\underline{r}, \tag{14}$$

and the complex envelope can be written as

$$\tilde{v}(t) = \omega_0 \int_{\text{object}} \left|\underline{B}_{r,xy}\left(\underline{r}\right)\right|\left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right| e^{-\imath\left[\Delta\omega\left(\underline{r}\right)t + \phi_{\text{pe}}\left(\underline{r}\right) - \phi_{\text{e}}\left(\underline{r}\right) + \phi_{\text{r}}\left(\underline{r}\right) - \frac{\pi}{2}\right]} d\underline{r}. \tag{15}$$

Note that both $\tilde{v}_R(t)$ and $\tilde{v}_I(t)$ are lowpass signals. In practice, the signals $\tilde{v}_R(t)$ and $\tilde{v}_I(t)$ can be obtained from the original signal $v(t)$ by multiplying $v(t)$ by a reference sinusoidal signal and then low-passing filtering to remove the high-frequency component. This method is known as the signal demodulation method, or the phase sensitive detection (PSD) method [15]. Using $2\cos(\omega_0 t)$ and $-2\sin(\omega_0 t)$ as reference signals, signal demodulation yields $\tilde{v}_R(t)$ and $\tilde{v}_I(t)$, respectively. This detection scheme is known as *quadrature detection*. Quadrature detection thus produces two data streams with a 90° phase difference. When put in complex form, with $\tilde{v}_R(t)$ being treated as the real part and $\tilde{v}_I(t)$ as the imaginary part, these data streams together constitute the complex envelope $\tilde{v}(t)$ of $v(t)$. Note that, given ω_0, all information content of $v(t)$ is preserved in the complex envelope $\tilde{v}(t)$.

To get more insight in the relation between $v(t)$ and its quadrature and in-phase components, consider the signal

$$v_+(t) = \tilde{v}(t)\exp(\imath\omega_0 t), \tag{16}$$

which is known as the *analytic signal* (or pre-envelope) of $v(t)$ and can be written as

$$v_+(t) = v(t) + \imath\breve{v}(t), \tag{17}$$

with $\breve{v}(t) = H[v(t)]$ [16], which will be denoted as $\breve{v}(t) = H[v(t)]$. In other words, $\breve{v}(t)$ is the response of a dynamic system with impulse response function

$$h(t) = \frac{1}{\pi t} \tag{18}$$

and corresponding frequency response function

$$H(\imath\omega) = -\imath\,\text{sgn}(\omega), \tag{19}$$

with $\text{sgn}(\omega)$ the sign of ω. The filter (19), which is known as a *quadratic filter* [16], has a constant amplitude $|H(\imath\omega)| = 1$ (all pass), and its phase equals $-\pi/2$ for $\omega > 0$ and $\pi/2$ for $\omega < 0$. The effect of forming the complex signal $v_+(t)$ is to remove the redundant negative frequency components of the Fourier transform. Indeed, it follows from above that the Fourier transform $\breve{V}(\omega)$ of $\breve{v}(t)$ is given by

$$\breve{V}(\omega) = -\imath\,\text{sgn}(\omega)V(\omega), \tag{20}$$

with $V(\omega)$ the Fourier transform of $v(t)$ and, as follows from Eqs. (17) and (20),

$$V_+(\omega) = V(\omega) + \text{sgn}(\omega)V(\omega). \tag{21}$$

Furthermore, the Fourier transform of the complex envelope $\tilde{v}(t)$ is given by

$$\tilde{V}(\omega) = V_+(\omega + \omega_0). \tag{22}$$

Using the Hilbert transform pairs $H[\cos(\omega t)] = \sin(\omega t)$ and $H[\sin(\omega t)] = -\cos(\omega t)$, and Bedrossian's theorem [20] it can be shown that:

$$\tilde{v}_R(t) = v(t)\cos(\omega_0 t) + \breve{v}(t)\sin(\omega_0 t), \tag{23}$$

$$\tilde{v}_I(t) = \breve{v}(t)\cos(\omega_0 t) - v(t)\sin(\omega_0 t). \tag{24}$$

Finally, if we assume that the receiver coil has a homogenous reception field, which may, for example, be a valid assumption in a single coil based on a birdcage volume resonator [15], the signal expression (15) can be further simplified to

$$\tilde{v}(t) = \int_{\text{object}} \underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right) e^{-\imath\left(\Delta\omega\left(\underline{r}\right)t + \phi_{\text{pe}}\left(\underline{r}\right)\right)} d\underline{r}, \tag{25}$$

where the complex notation

$$\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right) = \left|\underline{M}_{xy}\left(\underline{r},t_{\text{acq}}\right)\right| e^{\imath\phi_{\text{e}}\left(\underline{r}\right)}, \tag{26}$$

has been used and a scaling constant $=\omega_0 e^{\imath\pi/2}$ has been omitted [15]. Substituting Eqs. (3) and (5) in Eq. (25) then yields

$$\tilde{v}\left(\underline{k}\right) = \int_{\text{object}} \underline{M}_{xy}\left(\underline{r}, t_{\text{acq}}\right) e^{-\imath 2\pi \underline{k}\cdot\underline{r}} d\underline{r}, \tag{27}$$

where the mapping relation between (t, G_y) and $\underline{k} = (k_x, k_y)^T$ is given by

$$k_x = \frac{1}{2\pi}\gamma G_x t, \tag{28}$$

$$k_y = \frac{1}{2\pi}\gamma G_y T_{\text{pe}}. \tag{29}$$

Hence, the signal $\tilde{v}(\underline{k})$ in the so-called k-space is the 2D spatial Fourier transform of $\underline{M}_{xy}(\underline{r}, t_{\text{acq}})$, which is the quantity of interest to be reconstructed (i.e., the *image*). Note that if $\phi_e(\underline{r})$ is small or (ideally) zero, the imaginary component of $\underline{M}_{xy}(\underline{r}, t_{\text{acq}})$ can be neglected making the image to be reconstructed real valued. In practice, however, the realness constraint is often violated because object motion and magnetic field inhomogeneities introduce a nonzero phase to the image function [15]. Obviously, a straightforward reconstruction of the image is obtained by inverse Fourier transforming the data, but before we come to that, we will first consider the effect of noise. It should be noted that the assumption of a homogenous reception field is generally invalid for single-coil acquisitions that use so-called surface coils [21]. In that case, the detection sensitivity has to be taken into account and Eq. (15) can no longer be simplified to Eq. (25).

Modeling the noise

So far, we have considered noiseless signals. In practice, however, the signal $v(t)$ will be disturbed by noise, which is mainly caused by thermal motion (Brownian motion) of electrons within the body's conducting tissue and the receiving coil(s) [22]. This thermal noise, which was investigated experimentally by Johnson [23] and theoretically by Nyquist [24], is often referred to as Johnson noise. It can be modeled as additive zero mean white Gaussian noise with variance (or, power) [25,26]

$$\sigma_w^2 = 2k_b T\left(R_{\text{coil}} + R_{\text{body}}\right), \tag{30}$$

where k_b denotes Boltzmann's constant, T the absolute temperature and R_{coil} and R_{body} the effective resistance of the coil and the body, respectively.

Hence, the raw MR signal can be modeled as:

$$\boldsymbol{v}(t) = v(t) + \boldsymbol{n}_w(t), \tag{31}$$

with $\boldsymbol{n}_w(t)$ a stationary zero mean Gaussian white noise process with variance σ_w^2. Note that the signal $v(t)$ is band limited to frequencies ω such that $||\omega| - \omega_0| \le \max_{\underline{r}} \Delta\omega(\underline{r})$. The noise, however, has a spectrum that exists over the entire frequency range and can be separated into two components: (i) the out-of-band noise component $\mathbf{n}_o(t)$ and (ii) the in-band noise component $\mathbf{n}(t)$ [17]:

$$\mathbf{n}_w(t) = \mathbf{n}_o(t) + \mathbf{n}(t). \tag{32}$$

The in-band noise component $\mathbf{n}(t)$ can be obtained by filtering the raw data by an (ideal) bandpass filter with a passband that corresponds with the bandpass signal $v(t)$. Note that this filter leaves the bandpass signal $v(t)$ unaffected. Furthermore, note that $\mathbf{n}(t)$ is obtained from a Gaussian process through a linear operation. Hence, the process $\mathbf{n}(t)$ is also a Gaussian process. It is commonly known as a bandpass "white" Gaussian noise process, having a power spectral density function that is symmetrical about ω_0. It can be shown that $\mathbf{n}_0(t)$ is independent of both $v(t)$ and $\mathbf{n}(t)$ and can be discarded without any loss of information [17]. In what follows, we will assume that bandpass filtering has been applied to eliminate the out-of-band noise component.

Now, it can be shown that the band pass white Gaussian noise process $\mathbf{n}(t)$ can be written in the form [16]

$$\mathbf{n}(t) = \tilde{\boldsymbol{n}}_R(t)\cos(\omega_0 t) - \tilde{\boldsymbol{n}}_I(t)\sin(\omega_0 t), \tag{33}$$

with $\tilde{\boldsymbol{n}}_R(t)$ and $\tilde{\boldsymbol{n}}_I(t)$ zero mean lowpass stationary Gaussian processes described by

$$\tilde{\boldsymbol{n}}_R(t) = \mathbf{n}(t)\cos(\omega_0 t) + \breve{\boldsymbol{n}}(t)\sin(\omega_0 t), \tag{34}$$

$$\tilde{\boldsymbol{n}}_I(t) = \breve{\mathbf{n}}(t)\cos(\omega_0 t) - \boldsymbol{n}(t)\sin(\omega_0 t) \tag{35}$$

with $\breve{\boldsymbol{n}}(t)$ the Hilbert transform of $\boldsymbol{n}(t)$, where the Hilbert transform $\breve{\mathbf{x}}(t)$ of a stochastic process $\mathbf{x}(t)$ is given by the output of the system (19) with input $\mathbf{x}(t)$, that is [16]

$$\breve{\mathbf{x}}(t) = \frac{1}{\pi t} * \mathbf{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbf{x}(\tau)}{t-\tau} d\tau. \tag{36}$$

with $*$ the convolution operator. Note the analogy of Eqs. (34) and (35) with (23) and (24). It can easily be shown that since $\mathbf{n}(t)$ is a stationary process, the process $\breve{\mathbf{n}}(t)$ is also stationary [16]. Furthermore, it can be been shown that the following relations hold [16]:

$$R_{\breve{\mathbf{n}}\mathbf{n}}(0) = 0, \tag{37}$$

$$R_{\mathbf{n}}(\tau) = R_{\tilde{\boldsymbol{n}}_R}(\tau)\cos(\omega_0 \tau), \tag{38}$$

$$R_{\tilde{\boldsymbol{n}}_R}(\tau) = R_{\tilde{\boldsymbol{n}}_I}(\tau) = R_{\mathbf{n}}(\tau)\cos(\omega_0\tau) + R_{\breve{\mathbf{n}}\mathbf{n}}(\tau)\sin(\omega_0\tau), \tag{39}$$

$$R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}(\tau) = -R_{\tilde{\boldsymbol{n}}_I\tilde{\boldsymbol{n}}_R}(\tau) = 0, \quad \forall\tau \tag{40}$$

with $R_{\breve{\mathbf{n}}\mathbf{n}}(\tau)$ the cross-correlation function between the processes $\breve{\mathbf{n}}(t)$ and $\mathbf{n}(t)$, $R_{\mathbf{n}}(\tau)$ the autocorrelation function of the process $\mathbf{n}(t)$, $R_{\tilde{\boldsymbol{n}}_R}(\tau)$ the autocorrelation function of the process $\tilde{\boldsymbol{n}}_R(t)$, $R_{\tilde{\boldsymbol{n}}_I}(\tau)$ the autocorrelation function of the process $\tilde{\boldsymbol{n}}_I(t)$, and $R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}(\tau)$ the cross-correlation function of the processes $\tilde{\boldsymbol{n}}_R(t)$ and $\tilde{\boldsymbol{n}}_I(t)$. It follows from Eqs. (37) that, for a given t, the zero mean Gaussian random variables $\mathbf{n}(t)$ and $\breve{\mathbf{n}}(t)$ are orthogonal (and thus uncorrelated) and it follows from Eqs. (39) and (40) that the zero mean processes $\tilde{\boldsymbol{n}}_R(t)$ and $\tilde{\boldsymbol{n}}_I(t)$ have equal autocorrelation functions and are orthogonal.

In analogy with Eqs. (9) and (17), we can define the complex processes

$$\tilde{\mathbf{n}}(t) = \tilde{\boldsymbol{n}}_R(t) + \imath\tilde{\boldsymbol{n}}_I(t), \tag{41}$$

and

$$\mathbf{n}_+(t) = \mathbf{n}(t) + \imath\breve{\mathbf{n}}(t) = \tilde{\boldsymbol{n}}(t)\exp(\imath\omega_0 t), \tag{42}$$

representing the complex envelope and the analytic signal associated with $\boldsymbol{n}(t)$. Note that the last equality in Eq. (42) follows from Eqs. (34) and (35).

Modeling the noise disturbed MR signal

Let's now combine the results obtained in the Sections 2.1 and 2.2 and define

$$\boldsymbol{v}_+(t) = \tilde{\boldsymbol{v}}(t)\exp(\imath\omega_0 t), \tag{43}$$

with

$$\tilde{\boldsymbol{v}}(t) = \tilde{v}(t) + \tilde{\mathbf{n}}(t), \tag{44}$$

where $\tilde{v}(t)$ and $\tilde{\mathbf{n}}(t)$ are the previously defined *complex envelopes* of the signal $v(t)$ and the noise process $\mathbf{n}(t)$, respectively. The signal $\tilde{\boldsymbol{v}}(t)$ thus represents the complex envelope of the noise disturbed signal $v(t)+\mathbf{n}(t)$. It follows from above that $\tilde{\boldsymbol{v}}(t)$ can be described as

$$\tilde{\boldsymbol{v}}(t) = \tilde{\boldsymbol{v}}_R(t) + \imath\tilde{\boldsymbol{v}}_I(t), \tag{45}$$

with

$$\tilde{\boldsymbol{v}}_R(t) = \tilde{v}_R(t) + \tilde{\boldsymbol{n}}_R(t), \tag{46}$$

the noise disturbed in-phase component and

$$\tilde{\boldsymbol{v}}_I(t) = \tilde{v}_I(t) + \tilde{\boldsymbol{n}}_I(t), \tag{47}$$

the noise disturbed quadrature component. It can be shown that the signals $\tilde{\boldsymbol{v}}_R(t)$ and $\tilde{\boldsymbol{v}}_I(t)$ can be obtained by applying the signal demodulation method described in Section 2.1, the noise contributions to the in-phase and quadrature detection channel being equal to $\tilde{\boldsymbol{n}}_R(t)$ and $\tilde{\boldsymbol{n}}_I(t)$, respectively.

In summary, the noise disturbed MR data obtained by quadrature detection using a single receiver coil can be described in complex form as:

$$\tilde{\boldsymbol{v}}(t) = \tilde{v}_R(t) + \tilde{\boldsymbol{n}}_R(t) + \imath[\tilde{v}_I(t) + \tilde{\boldsymbol{n}}_I(t)]. \tag{48}$$

where $\tilde{v}_R(t)$ and $\tilde{v}_I(t)$ are the in-phase (or, real) and quadrature (or, imaginary) components of the noiseless signal and $\tilde{\boldsymbol{n}}_R(t)$ and $\tilde{\boldsymbol{n}}_I(t)$ are two zero mean Gaussian, orthogonal processes that describe the in-phase and quadrature component of the noise, respectively.

Sampling

In MRI practice, the signal Eq. (48) will be sampled, and sampling may affect the correlation properties of the noise. Recall that the noise process $\mathbf{n}(t)$ is assumed to be the result of bandpass filtering a continuous time white Gaussian noise process $\boldsymbol{n}_W(t)$ over a band centered around ω_0, where the width W of the band corresponds with the passband of the bandpass signal $v(t)$. Furthermore, recall that the power spectral density function (and thus variance) of $\boldsymbol{n}_W(t)$ was equal to σ_W^2. Then, the autocorrelation function of the bandpass "white" noise process $\mathbf{n}(t)$ is given by Ref. [19]

$$R_{\boldsymbol{n}}(\tau) = 2\sigma_W^2 \frac{\sin\left(\frac{W}{2}\tau\right)}{\pi\tau}\cos(\omega_0\tau). \tag{49}$$

By comparison with Eq. (38) we have

$$R_{\tilde{\boldsymbol{n}}_R}(\tau) = R_{\tilde{\boldsymbol{n}}_I}(\tau) = 2\sigma_W^2 \frac{\sin\left(\frac{W}{2}\tau\right)}{\pi\tau}. \tag{50}$$

If we sample the complex envelope $\tilde{\mathbf{n}}(t) = \tilde{\boldsymbol{n}}_R(t) + \imath\tilde{\boldsymbol{n}}_I(t)$ at the Nyquist rate of $\omega_s = 2\pi/\Delta t = W$, we have:

$$R_{\tilde{\boldsymbol{n}}_R}(l\Delta t) = R_{\tilde{\boldsymbol{n}}_I}(l\Delta t) = \frac{\sigma_W^2 W}{\pi}\frac{\sin(\pi l)}{\pi l} = \frac{\sigma_W^2 W}{\pi}\delta[l], \tag{51}$$

with $l \in \mathbb{Z}$ and $\delta[\cdot]$ the Kronecker delta function. Furthermore, as derived earlier,

$$R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}(\tau) = -R_{\tilde{\boldsymbol{n}}_I\tilde{\boldsymbol{n}}_R}(\tau) = 0. \tag{52}$$

Hence,

$$R_{\tilde{\boldsymbol{n}}_R}[l] = R_{\tilde{\boldsymbol{n}}_I}[l] = \frac{\sigma_W^2 W}{\pi}\delta[l], \tag{53}$$

and

$$R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}[l] = -R_{\tilde{\boldsymbol{n}}_I\tilde{\boldsymbol{n}}_R}[l] = 0, \tag{54}$$

where $R_{\tilde{\boldsymbol{n}}_R}[l] = R_{\tilde{\boldsymbol{n}}_R}(l\Delta t)$, $R_{\tilde{\boldsymbol{n}}_I}[l] = R_{\tilde{\boldsymbol{n}}_I}(l\Delta t)$, $R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}[l] = R_{\tilde{\boldsymbol{n}}_R\tilde{\boldsymbol{n}}_I}(l\Delta t)$ and $R_{\tilde{\boldsymbol{n}}_I\tilde{\boldsymbol{n}}_R}[l] = R_{\tilde{\boldsymbol{n}}_I\tilde{\boldsymbol{n}}_R}(l\Delta t)$. Hence, the discrete random process obtained by sampling $\tilde{\mathbf{n}}(t)$ at the Nyquist rate is complex white Gaussian noise [19].

Its worthwhile mentioning that, assuming frequency encoding along the x-direction, it follows from Eq. (3) that the bandwidth W is directly related to the field of view in the x-direction:

$$W = 2\max_{\underline{r}}|\Delta\omega(\underline{r})| = 2\gamma\max_x|G_x x| = \gamma|G_x|\text{FOV}_x, \tag{55}$$

with FOV_x the field of view in the x-direction.

Modeling k-space data

It follows from Eqs. (25), (28) and (29) that, for a given value of G_y, sampling the signal $\tilde{\boldsymbol{v}}(t)$ at sample intervals Δt corresponds with sampling the k-space along a line of constant k_y at sample intervals $\Delta k_x = (1/2\pi)\gamma G_x \Delta t$. Assuming a rectilinear sampling scheme, $\tilde{\boldsymbol{v}}(t, G_y)$ is sampled line by line, and the two-dimensional sampling problem can be treated along each dimension separately. Using the mapping relations (28) and (29), it can be shown that the largest sampling intervals permissible by the Nyquist criterion are

$$\Delta t = \frac{2\pi}{\gamma|G_x|\text{FOV}_x} \tag{56}$$

and

$$\Delta G_y = \frac{2\pi}{\gamma T_{pe}\text{FOV}_y}, \tag{57}$$

with FOV_y the field of view in the y-direction [15].

Finally, let $\boldsymbol{z}(\underline{k})$ denote the signal $\tilde{\boldsymbol{v}}(t, G_y)$ that has been mapped to the k-space (i.e., spatial frequency space), using the mapping relations (28) and (29):

$$\boldsymbol{z}\left(\underline{k}\right) \equiv \tilde{\boldsymbol{v}}\left(\underline{k}\right) = \tilde{v}_R\left(\underline{k}\right) + \tilde{\boldsymbol{n}}_R\left(\underline{k}\right) + \imath\left[\tilde{v}_I\left(\underline{k}\right) + \tilde{\boldsymbol{n}}_I\left(\underline{k}\right)\right]. \tag{58}$$

Furthermore, assume that the k-space has been Nyquist sampled in the sample points $\underline{k}_1, \ldots, \underline{k}_N$ and define the complex random vector

$$\underline{\mathbf{z}} = \begin{pmatrix} \mathbf{z}\left(\underline{k}_1\right) \\ \vdots \\ \mathbf{z}\left(\underline{k}_N\right) \end{pmatrix} \tag{59}$$

and the complex deterministic vector

$$\underline{s} = \begin{pmatrix} s\left(\underline{k}_1\right) \\ \vdots \\ s\left(\underline{k}_N\right) \end{pmatrix}, \tag{60}$$

with $s(\underline{k}_i) = s_R(\underline{k}_i) + \imath s_I(\underline{k}_i)$, for i=1,…,N. It follows from the analysis described in Section 2.4 that the real and imaginary parts $\mathbf{z}_{\mathrm{R}}(\underline{k}_i)$ and $\mathbf{z}_{\mathrm{I}}(\underline{k}_i)$ of the complex random variables $\mathbf{z}(\underline{k}_i)$ are independent, Gaussian distributed with equal variance $\sigma_K^2 = \sigma_{\mathrm{w}}^2 W/\pi$, which implies that

$$\mathrm{cov}\left(\mathbf{z}_{\mathrm{R}}\left(\underline{k}_i\right), \mathbf{z}_{\mathrm{I}}\left(\underline{k}_j\right)\right) = 0, \forall i, j \tag{61}$$

and

$$\mathrm{cov}\left(\mathbf{z}_{\mathrm{R}}\left(\underline{k}_i\right), \mathbf{z}_{\mathrm{R}}\left(\underline{k}_j\right)\right) = \mathrm{cov}\left(\mathbf{z}_{\mathrm{I}}\left(\underline{k}_i\right), \mathbf{z}_{\mathrm{I}}\left(\underline{k}_j\right)\right) = \sigma_K^2 \delta[i-j]. \tag{62}$$

Using basic theory on complex Gaussian distributions (see Appendix A), it then follows that the joint probability density function of the complex random variable $\underline{\mathbf{z}}$ is given by:

$$f_{\underline{\mathbf{z}}}(\underline{z}) = \frac{1}{\pi^N \det(\Sigma_z)} \exp\{-(\underline{z}-\underline{s})^H \Sigma_z^{-1} (\underline{z}-\underline{s})\}, \tag{63}$$

with $\Sigma_z = 2\sigma_K^2 I_N$, where I_N is the identity matrix of order N. This is usually denoted by

$$\underline{\mathbf{z}} \sim \mathcal{CN}\left(\underline{s}, 2\sigma_K^2 I_N\right). \tag{64}$$

This PDF is called the joint circularly complex normal distribution, also known as the complex multivariate normal (or Gaussian) PDF [19,27].

Expression (64) is the main result of this section and forms the starting point of Section 3, in which we will analyze how the data distribution changes when $\underline{\mathbf{z}}$ is further processed for the purpose of image reconstruction.

MRI data distributions of single-coil images

From the complex data in the k-space (i.e., the spatial frequency domain) a so-called reconstructed MR image in the spatial domain can be obtained by taking the inverse two-dimensional (2D) Discrete Fourier Transform (DFT). From the complex valued reconstructed image thus obtained, magnitude and phase images can be created straightforwardly. As illustrated in Section 2, the pixel values of the magnitude image are directly related to the strength of the transverse component of the net transverse magnetization in the volume elements, voxels, in the selected tissue slice. The phase image is often discarded since it may exhibit incidental phase variations due to RF angle inhomogeneity, filter responses, system delay, noncentred sampling windows, a time-varying behavior due to radio frequency angle inhomogeneity, system delay, field inhomogeneities, chemical shift, etc. [28–30]. Magnitude images are immune to these effects. Nevertheless, phase images may contain valuable information. For example, phase images are used to measure flow [31–35] or susceptibility [36–40].

In this section, we will describe the distribution of reconstructed complex, magnitude and phase images acquired with a single-coil acquisition system. The distribution of images acquired with multiple-coils systems will be described in Section 4.

Statistical distribution of single-coil complex images

As was derived in Section 2, the complex data acquired in k-space, often referred to as the *raw data*, can (pixelwise) be described as

$$\boldsymbol{z}\left(\underline{k}\right) = s\left(\underline{k}\right) + \boldsymbol{n}\left(\underline{k}\right), \tag{65}$$

with $s(\underline{k})$ the complex noise free data and $\boldsymbol{n}(\underline{k})$ additive noise that can be modeled as a zero mean circular complex Gaussian random variable (see Appendix A), whose PDF is given by

$$f_{\boldsymbol{n}\left(\underline{k}\right)}\left(n\left(\underline{k}\right)\right) = \frac{1}{2\pi\sigma_K^2} \exp\left(-\left|n\left(\underline{k}\right)\right|^2 / \left(2\sigma_K^2\right)\right). \tag{66}$$

This is usually denoted as

$$\boldsymbol{n}\left(\underline{k}\right) \sim \mathcal{CN}\left(0, 2\sigma_K^2\right), \tag{67}$$

which implies that the real and imaginary components $\boldsymbol{n}_{\mathrm{R}}(\underline{k})$ and $\boldsymbol{n}_{\mathrm{I}}(\underline{k})$ of $\mathbf{n}(\underline{k})$ are independent identically distributed (i.i.d.) zero-mean Gaussian random variables with variance σ_K^2 (see Appendix A). That is,

$$\boldsymbol{n}_{\mathrm{R}}\left(\underline{k}\right) \sim \mathcal{N}\left(0, \sigma_K^2\right), \tag{68}$$

$$\boldsymbol{n}_{\mathrm{I}}\left(\underline{k}\right) \sim \mathcal{N}\left(0, \sigma_K^2\right), \tag{69}$$

and $\mathrm{cov}(\boldsymbol{n}_{\mathrm{R}}(\underline{k}), \boldsymbol{n}_{\mathrm{I}}(\underline{k})) = 0$. Moreover, it follows from Eq. (66) that the RV $\mathbf{n}(\underline{k})$ has the same distribution as the RV $e^{\imath\theta}\mathbf{n}(\underline{k}), \forall\theta\in\mathbb{R}$. The RV $\mathbf{n}(\underline{k})$ is therefore called *circularly symmetric* [41].

Furthermore, as explained in Section 2, the noise process $\boldsymbol{n}(\underline{k})$ can be assumed to be stationary so that σ_K^2 does not depend on $\underline{k}$. Moreover, as was shown in Section 2, if we assume that the k-space was sampled at the Nyquist rate, the complex Gaussian distributed sample points in the k-space are uncorrelated (and therefore independent).

Next, a complex image in the spatial domain (or, *image space*) is obtained by taking the inverse DFT of the complex data in the k-space. Due to the linearity and orthogonality of the Fourier transform, the complex data points in the image space are also independent Gaussian distributed, as is illustrated in Appendix B. Hence, the complex image in the spatial domain can (pixelwise) be modeled as

$$\boldsymbol{z}\left(\underline{r}\right) = s\left(\underline{r}\right) + \boldsymbol{n}\left(\underline{r}\right), \tag{70}$$

with

$$s\left(\underline{r}\right) = s_{\mathrm{R}}\left(\underline{r}\right) + \imath s_{\mathrm{I}}\left(\underline{r}\right), \tag{71}$$

the noise free signal and

$$\boldsymbol{n}\left(\underline{r}\right) = \boldsymbol{n}_{\mathrm{R}}\left(\underline{r}\right) + \imath\boldsymbol{n}_{\mathrm{I}}\left(\underline{r}\right) \tag{72}$$

the additive noise contribution, and $\boldsymbol{n}(\underline{r}) \sim \mathcal{CN}(0,\ 2\sigma^2)$, with $\sigma^2 = (1/N)\sigma_K^2$, where N is the number of points used to compute the inverse DFT (see Appendix B). This implies that

$$\boldsymbol{n}_{\mathrm{R}}\left(\underline{r}\right) \sim \mathcal{N}\left(0, \sigma^2\right), \tag{73}$$

$$\boldsymbol{n}_{\mathrm{I}}\left(\underline{r}\right) \sim \mathcal{N}\left(0, \sigma^2\right), \tag{74}$$

and

$$\mathrm{cov}\left(\boldsymbol{n}_{\mathrm{R}}\left(\underline{r}\right), \boldsymbol{n}_{\mathrm{I}}\left(\underline{r}\right)\right) = 0. \tag{75}$$

The probability density function (PDF) of the complex Gaussian RV $\boldsymbol{z}(\underline{r})$ is then given by

$$f_{\boldsymbol{z}\left(\underline{r}\right)}\left(z\left(\underline{r}\right)\right) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\left|z\left(\underline{r}\right) - s\left(\underline{r}\right)\right|^2}{2\sigma^2}\right), \tag{76}$$

which is usually denoted as $\boldsymbol{z}(\underline{r}) \sim \mathcal{CN}(s(\underline{r}),\ 2\sigma^2)$.

Statistical distribution of single-coil magnitude images

A magnitude image is obtained by taking (pixel by pixel) the root sum of squares (SoS) of the real and imaginary part of the complex image $\boldsymbol{z}(\underline{r})$:

$$\mathbf{m}(\underline{r}) = \sqrt{\boldsymbol{z}_{\mathrm{R}}^2(\underline{r}) + \boldsymbol{z}_{\mathrm{I}}^2(\underline{r})} = \sqrt{|\boldsymbol{z}(\underline{r})|^2}. \tag{77}$$

For notational convenience, we will suppose that all the equations are pixelwise and write $\boldsymbol{z}$ and $\boldsymbol{m}$ instead of $\boldsymbol{z}(\underline{r})$ and $\boldsymbol{m}(\underline{r})$. It can be shown that the random variable $\mathbf{m}$ is Rician distributed. Its PDF $f_{\mathbf{m}}(m)$ is given by Ref. [42]

$$f_{\mathbf{m}}(m) = \frac{m}{\sigma^2}e^{-\frac{a^2+m^2}{2\sigma^2}}I_0\left(\frac{ma}{\sigma^2}\right)\varepsilon(m), \tag{78}$$

where $I_0(\cdot)$ is the 0th order modified Bessel function of the first kind and $a^2 = s_{\mathrm{R}}^2 + s_{\mathrm{I}}^2$, with s_{R} and s_{I} the real and imaginary part of $s = \mathbb{E}[\mathbf{z}]$. The unit step Heaviside function $\varepsilon(.)$ is used to indicate that the expression for the PDF of $\mathbf{m}$ is valid for non-negative values of m only.

The shape of the PDF (78) depends on the Signal to Noise Ratio (SNR), which we will define as a/σ. In the special case $a = 0$ (no signal, SNR = 0), the Rician PDF turns into a Rayleigh PDF given by Ref. [43,44]

$$f_{\mathbf{m}}(m) = \frac{m}{\sigma^2}e^{-\frac{m^2}{2\sigma^2}}\varepsilon(m). \tag{79}$$

For increasing values of the SNR, that is, for SNR $\rightarrow \infty$, the asymptotic expansion of $I_0(x)$ when x is large is [45]

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{1}{8x} + \frac{1\cdot 9}{2!\cdot(8x)^2} + \frac{1\cdot 9\cdot 25}{3!\cdot(8x)^3} + \ldots\right]. \tag{80}$$

Then, for sufficiently large x, $I_0(x) \approx e^x/\sqrt{2\pi x}$ and the Rician distribution (78) can be approximated as follows:

$$f_{\mathbf{m}}(m) = \sqrt{\frac{m}{2\pi\sigma^2 a}}\exp\left(-\frac{(m-a)^2}{2\sigma^2}\right), \tag{81}$$

or even further by a Gaussian distribution with corresponding PDF

$$f_{\mathbf{m}}(m) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(m-a)^2}{2\sigma^2}\right). \tag{82}$$

The moments (or raw moments) of the Rician distribution can be expressed analytically as [46]

$$\mathbb{E}[\mathbf{m}^r] = \left(2\sigma^2\right)^{r/2}\Gamma\left(1 + \frac{r}{2}\right){}_1F_1\left[-\frac{r}{2}; 1; -\frac{a^2}{2\sigma^2}\right], \tag{83}$$

where $\Gamma(\cdot)$ represents the Gamma function and ${}_1F_1(\cdot;\cdot;\cdot)$ denotes the confluent hypergeometric function of the first kind. The first four moments of the Rice PDF are given by Ref. [47]

$$\mathbb{E}[\mathbf{m}] = \sigma\sqrt{\frac{\pi}{2}}e^{-\frac{a^2}{4\sigma^2}}\left[\left(1 + \frac{a^2}{2\sigma^2}\right)I_0\left(\frac{a^2}{4\sigma^2}\right) + \frac{a^2}{2\sigma^2}I_1\left(\frac{a^2}{4\sigma^2}\right)\right], \tag{84}$$

$$\mathbb{E}\left[\mathbf{m}^2\right] = a^2 + 2\sigma^2, \tag{85}$$

$$\mathbb{E}\left[\mathbf{m}^3\right] = \sigma^3\sqrt{\frac{\pi}{2}}e^{-\frac{a^2}{4\sigma^2}}\left[\left(3 + 3\frac{a^2}{\sigma^2} + \frac{a^4}{2\sigma^4}\right)I_0\left(\frac{a^2}{4\sigma^2}\right) + \left(2\frac{a^2}{\sigma^2} + \frac{a^4}{2\sigma^4}\right)I_1\left(\frac{a^2}{4\sigma^2}\right)\right], \tag{86}$$

$$\mathbb{E}\left[\mathbf{m}^4\right] = a^4 + 8\sigma^2a^2 + 8\sigma^4, \tag{87}$$

with $I_1(\cdot)$ denoting the 1st order modified Bessel function of the first kind. Note that the even moments are simple polynomials. The expressions for the odd moments are more complex and have been derived using the fact that the confluent hypergeometric function can be expressed in terms of modified Bessel functions [45]. The variance of the Rician distributed RV $\mathbf{m}$ is given by

$$\begin{aligned} Var(\boldsymbol{m}) &= \mathbb{E}\left[\boldsymbol{m}^2\right] - \mathbb{E}[\boldsymbol{m}]^2 \\ &= a^2 + 2\sigma^2 \\ &\quad - \frac{\pi\sigma^2}{2}e^{-\frac{a^2}{2\sigma^2}}\left[\left(1 + \frac{a^2}{2\sigma^2}\right)I_0\left(\frac{a^2}{4\sigma^2}\right) + \frac{a^2}{2\sigma^2}I_1\left(\frac{a^2}{4\sigma^2}\right)\right]^2. \end{aligned} \tag{88}$$

Since the Rayleigh PDF is a special case of the Rice PDF (with $a = 0$), expressions for its moments can be directly derived from the expressions above, yielding [48]

$$\mathbb{E}[\mathbf{m}^r] = \left(2\sigma^2\right)^{r/2}\Gamma\left(1 + \frac{r}{2}\right), \tag{89}$$

or explicitly for the first four moments:

$$\mathbb{E}[\mathbf{m}] = \sqrt{\frac{\pi}{2}}\sigma, \tag{90}$$

$$\mathbb{E}\left[\mathbf{m}^2\right] = 2\sigma^2, \tag{91}$$

$$\mathbb{E}\left[\mathbf{m}^3\right] = 3\sqrt{\frac{\pi}{2}}\sigma^3, \tag{92}$$

$$\mathbb{E}\left[\mathbf{m}^4\right] = 8\sigma^4, \tag{93}$$

and the variance (88) simplifies to

$$Var(\mathbf{m}) = \sigma^2\left(2 - \frac{\pi}{2}\right). \tag{94}$$

At high SNR, on the other hand, the Rician PDF tends to a Gaussian PDF and (88) will reduce to the simple expression

$$Var(\mathbf{m}) = \sigma^2. \tag{95}$$

3.3. Statistical distribution of single-coil phase images

Recall from Section 3.1 that the PDF of the complex Gaussian RV $\boldsymbol{z}$ with mean s and variance $2\sigma^2$ is given by

$$f_{\boldsymbol{z}}(z) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|z-s|^2}{2\sigma^2}\right), \tag{96}$$

which is usually denoted as $\boldsymbol{z} \sim \mathcal{CN}(s,\ 2\sigma^2)$. If we write $\mathbf{z}$ in polar coordinates, we get $\mathbf{z} = \boldsymbol{m}e^{\imath\boldsymbol{\theta}}$, where the real valued RVs $\boldsymbol{m}$ and $\boldsymbol{\theta}$ denote the magnitude and phase of $\mathbf{z}$, respectively. Similarly, the complex mean s can also be written in polar coordinates: $s = ae^{\imath\phi}$. The joint PDF of $\mathbf{m}$ and $\boldsymbol{\theta}$ is obtained by rewriting Eq. (96) as

$$f_{\mathbf{m},\theta}(m,\theta) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|me^{\imath\theta}-ae^{\imath\phi}|^2}{2\sigma^2}\right). \tag{97}$$

As discussed above, the magnitude $\mathbf{m}$ is Rician distributed. Its PDF is given by (78). The conditional distribution of the phase $\boldsymbol{\theta}$, given the magnitude, follows a so-called Tikhonov distribution [49,50]:

$$f_{\boldsymbol{\theta}|\boldsymbol{m}}(\theta|m) = \frac{\exp[\lambda\cos(\theta-\phi)]}{2\pi I_0(\lambda)}, \tag{98}$$

with $\lambda = ma/\sigma^2$. It is obtained by dividing Eq. (97) by Eq. (78). The marginal PDF of the phase $\boldsymbol{\theta}$ is obtained by integrating (97) over m yielding [51–53]

$$f_{\boldsymbol{\theta}}(\theta) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2\right] \times \left[1+\kappa\sqrt{\pi}\exp\left(\kappa^2\right)(1+\mathrm{erf}(\kappa))\right], \tag{99}$$

with erf(•) the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \tag{100}$$

and

$$\kappa = \frac{1}{\sqrt{2}}\frac{a}{\sigma}\cos(\theta-\phi). \tag{101}$$

Note that, unlike $\boldsymbol{z}_R$ and $\boldsymbol{z}_I$, the RVs $\mathbf{m}$ and $\boldsymbol{\theta}$ are generally not independent. However, when $a = 0$, Eqs. (98) and (99) reduce to a uniform PDF $f_{\boldsymbol{\theta}|\boldsymbol{m}}(\theta|m) = f_{\boldsymbol{\theta}}(\theta) = 1/2\pi$ and $\mathbf{m}$ and $\boldsymbol{\theta}$ are independent. For high SNR on the other hand, Eq. (99) tends to the Gaussian PDF [53]

$$f_{\boldsymbol{\theta}}(\theta) = \frac{1}{\sqrt{2\pi}}\frac{a}{\sigma}\exp\left[-\frac{a^2(\theta-\phi)^2}{2\sigma^2}\right]. \tag{102}$$

3.4. Simplifications at high SNR

Note that in polar coordinates, the complex images can (pixel-wise) be described as

$$\mathbf{z}\left(\underline{r}\right) = \mathbf{m}\left(\underline{r}\right)e^{\imath\theta\left(\underline{r}\right)} = a\left(\underline{r}\right)e^{\imath\phi\left(\underline{r}\right)} + \left|\boldsymbol{n}\left(\underline{r}\right)\right|e^{\imath\psi\left(\underline{r}\right)} \tag{103}$$

with

$$\left|\boldsymbol{n}\left(\underline{r}\right)\right| = \sqrt{\boldsymbol{n}_R^2\left(\underline{r}\right)+\boldsymbol{n}_I^2\left(\underline{r}\right)}, \tag{104}$$

the Rayleigh distributed magnitude of the noise and

$$\psi\left(\underline{r}\right) = \tan^{-1}\left(\frac{\boldsymbol{n}_I\left(\underline{r}\right)}{\boldsymbol{n}_R\left(\underline{r}\right)}\right), \tag{105}$$

the uniformly distributed phase of the noise. The random variables $\left|\boldsymbol{n}(\underline{r})\right|$ and $\psi(\underline{r})$ are independent and their statistical properties do not depend on $\underline{r}$. From now on, the dependence of $\mathbf{z}$, $\mathbf{m}$, a, ϕ, $\boldsymbol{n}$ and ψ on $\underline{r}$ is assumed but dropped from the notation. Expression (103) can be rewritten as

$$\begin{aligned}\mathbf{z} &= e^{\imath\phi}\left(a+|\boldsymbol{n}|e^{\imath(\psi-\phi)}\right)\\ &= e^{\imath\phi}(a+|\mathbf{n}|\cos(\psi-\phi)+\imath|\mathbf{n}|\sin(\psi-\phi)),\end{aligned} \tag{106}$$

with $|\mathbf{n}|\cos(\psi-\phi)$ the noise component that is collinear (i.e., in phase) with the signal and $|\mathbf{n}|\sin(\psi-\phi)$ the noise component that is out of phase with the signal. It follows from Eq. (106) that

$$\boldsymbol{m} \equiv |\mathbf{z}| = |a+|\mathbf{n}|\cos(\psi-\phi)+\imath|\boldsymbol{n}|\sin(\psi-\phi)|. \tag{107}$$

Starting from Eq. (103) and assuming that ψ is distributed uniformly, independent of the value of $|\boldsymbol{n}|$, Hayes and Roemer [54] analyzed the variance of $\boldsymbol{m}$ at high SNR by replacing squares and square roots by power series expansions to second order in $|\boldsymbol{n}|/a$, resulting in

$$Var(\mathbf{m}) \approxeq \frac{\mathbb{E}\left[|\boldsymbol{n}|^2\right]}{2}. \tag{108}$$

Hayes and Roemer [54] found that the approximation (108) is equivalent to ignoring the noise that is out of phase with the signal, that is, by ignoring the imaginary part of the last term in (106). Indeed it can be shown straightforwardly that

$$Var(a+|\mathbf{n}|\cos(\psi-\phi)) = \frac{\mathbb{E}\left[|\boldsymbol{n}|^2\right]}{2}. \tag{109}$$

Hence, one may conclude that in the high SNR case (i.e., $SNR \geq 10$), only the noise in-phase with the signal will contribute to the magnitude image, whereas the contribution of the out-of phase noise can be neglected [54,55].

Assuming $\mathbf{n} \sim \mathcal{CN}(0, 2\sigma^2)$, $|\boldsymbol{n}|$ is Rayleigh distributed and (108) reduces to

$$Var(\mathbf{m}) \approxeq \frac{2\sigma^2}{2} = \sigma^2. \tag{110}$$

Note that this observation is in agreement with the earlier mentioned result that for high SNR the distribution of $\mathbf{m}$ tends to $\mathcal{N}(a,\sigma^2)$. Furthermore, remark that if the signal $ae^{\imath\phi}$ to be reconstructed (i.e., the *image*) is known to be real valued and positive (i.e., $\phi = 0$), expression (107) reduces to

$$\boldsymbol{m} = |\mathbf{z}| = |a+|\mathbf{n}|\cos(\psi)+\imath|\mathbf{n}|\sin(\psi)|, \tag{111}$$

which can be rewritten as

$$\boldsymbol{m} = |\mathbf{z}| = |a+\mathbf{n}_R+\imath\mathbf{n}_I| \tag{112}$$

In this case, the out of phase noise component corresponds with the imaginary part of the noise. It now follows from the analysis above, that for high SNR and $\phi = 0$, one may reasonably assume that one is dealing solely with the real part of the noise as is often practiced [55]. Note that this assumption will no longer be valid at low(er) SNR. Furthermore, the realness assumption is often violated because object motion and magnetic field inhomogeneities introduce a nonzero phase ϕ to the images.

4. Statistical distribution of multiple-coil images

So far, we have assumed that the images are acquired with a single receiver coil. However, image acquisition with multiple coils is becoming more and more common nowadays. Therefore, this section considers the distribution of MR images acquired by multiple-coil systems.

Before we continue, it should be mentioned that parallel MRI (pMRI) methods are outside the scope of this paper. pMRI allows reducing the acquisition time by subsampling the *k*-space, at the expense of aliasing and other artifacts in the image space. As a consequence, SoS can no longer be used as reconstruction method. Reconstruction methods such as sensitivity encoding (SENSE) and GeneRalized Autocalibrated Partially Parallel Acquisition (GRAPPA) have been introduced to suppress or correct these artifacts. For a review of pMRI methods, the reader is referred to [56]. Furthermore, for an analysis of the noise in GRAPPA and SENSE reconstructed images, see Ref. [57] and Ref. [58], respectively. Generally, the distribution of images acquired by pMRI methods is still a subject of current research.

4.1. Statistical distribution of multiple-coil complex images

When images are acquired with multiple (say L) receiver coils, the *k*-space is effectively sampled L times, resulting in L sets of complex raw data. Taking the inverse DFT of each of these data sets results in L complex images in the image space. We have seen above, that the pixels of each of these complex images can be modeled as circularly complex Gaussian random variables (i.e., as complex random variables whose real and imaginary parts are independent, Gaussian distributed with equal variance):

$$\mathbf{z}_l\left(\underline{r}\right) \sim \mathcal{CN}\left(s_l\left(\underline{r}\right), 2\sigma_l^2\right), \quad l = 1, \ldots, L \tag{113}$$

with $s_l(\underline{r})$ the expected value and $2\sigma_l^2$ the variance of the pixels of the complex image acquired with the lth coil. Note that the variance of the real and imaginary part of $\mathbf{z}_l(\underline{r})$ is given by σ_l^2. Define (for each $\underline{r}$) the complex random vector

$$\underline{\mathbf{z}}\left(\underline{r}\right) = \begin{pmatrix} \mathbf{z}_1\left(\underline{r}\right) \\ \vdots \\ \mathbf{z}_L\left(\underline{r}\right) \end{pmatrix}, \tag{114}$$

and the complex deterministic vector

$$\underline{s}\left(\underline{r}\right) = \begin{pmatrix} s_1\left(\underline{r}\right) \\ \vdots \\ s_L\left(\underline{r}\right) \end{pmatrix}. \tag{115}$$

In what follows, we will write $\underline{\mathbf{z}}$ and $\underline{s}$ instead of $\underline{\mathbf{z}}(\underline{r})$ and $\underline{s}(\underline{r})$ to simplify the notation. Next, let $\underline{\mathbf{z}}_\mathrm{R}$ denote the real part and $\underline{\mathbf{z}}_\mathrm{I}$ the imaginary part of $\underline{\mathbf{z}}$ and define

$$\Sigma_R = \mathrm{cov}(\underline{\mathbf{z}}_\mathrm{R}, \underline{\mathbf{z}}_\mathrm{R}), \tag{116}$$

$$\Sigma_I = \mathrm{cov}(\underline{\mathbf{z}}_\mathrm{I}, \underline{\mathbf{z}}_\mathrm{I}), \tag{117}$$

$$\Sigma_{IR} = \Sigma_{RI}^T = \mathrm{cov}(\underline{\mathbf{z}}_\mathrm{I}, \underline{\mathbf{z}}_\mathrm{R}), \tag{118}$$

and

$$\underline{\boldsymbol{w}} = \begin{pmatrix} \underline{\mathbf{z}} \\ \underline{\mathbf{z}}^* \end{pmatrix}. \tag{119}$$

Then it can be shown that the PDF of $\underline{\mathbf{z}}$ is given by (see Appendix A) [59]

$$\frac{1}{\pi^L \sqrt{\det(\Sigma_w)}} \exp\left\{ -\frac{1}{2}\left(\underline{w} - \mathbb{E}\left[\underline{\mathbf{w}}\right]\right)^H \Sigma_w^{-1} \left(\underline{w} - \mathbb{E}\left[\underline{\mathbf{w}}\right]\right)\right\} \tag{120}$$

where the elements of $\underline{w}$ correspond with those of $\underline{\mathbf{w}}$ and $\Sigma_w = \mathrm{cov}(\underline{\mathbf{w}}, \underline{\mathbf{w}})$.

Next, suppose that we can assume that the correlation structure of the real parts of the noise at the different coils is equal to the correlation structure of the imaginary parts, that is,

$$\Sigma_R = \Sigma_I. \tag{121}$$

Furthermore, let's suppose that we can assume that

$$\Sigma_{IR} = -\Sigma_{IR}^T. \tag{122}$$

If condition (121) and condition (122) are both satisfied, Eq. (120) simplifies to a so-called joint circularly complex normal distribution [59], also known as the complex multivariate Gaussian PDF [19] (see Appendix A):

$$f_{\underline{\mathbf{z}}}(\underline{z}) = \frac{1}{\pi^L \det(\Sigma_z)} \exp\{ -(\underline{z} - \underline{s})^H \Sigma_z^{-1} (\underline{z} - \underline{s})\}, \tag{123}$$

where the elements of the vector $\underline{z}$ correspond with those of $\underline{\mathbf{z}}$ and $\Sigma_z = \mathrm{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}) = 2\Sigma_R + 2\imath\Sigma_{IR}$. This is usually denoted as $\underline{\mathbf{z}} \sim \mathcal{CN}(\underline{s}, \Sigma_z)$.

Note that condition (122) implies that Σ_{IR} has a zero main diagonal. This means that the real and imaginary part of each component $\mathbf{z}_k$ of $\underline{\mathbf{z}}$ are uncorrelated, which is a valid assumption, as was derived in Section 2. Furthermore, a sufficient, but not necessary, condition for (122) to be satisfied is

$$\Sigma_{IR} = O, \tag{124}$$

with O the $L \times L$ null matrix. In that case, the real part of $\mathbf{z}_k$ and the imaginary part of $\mathbf{z}_l$ are uncorrelated not only for $k = l$, but also for $k \neq l$. This seems to be a reasonable assumption that is often (implicitly) practiced [57,60].

Moreover, if we not only assume that conditions (121), (122) and (124) are satisfied, but additionally assume that there is no correlation between the coils and that the variance of the noise at each coil is the same, then Σ_R and Σ_I will be diagonal matrices with identical eigenvalues:

$$\Sigma_R = \Sigma_I = \sigma^2 I_L, \tag{125}$$

where I_L is the identity matrix of order L. In this case,

$$\Sigma_z = 2\sigma^2 I_L \tag{126}$$

and Eq. (123) further simplifies to

$$f_{\underline{\mathbf{z}}}(\underline{z}) = \frac{1}{\pi^L \det(\Sigma_z)} \exp\{-\frac{1}{2\sigma^2}(\underline{z}-\underline{s})^H(\underline{z}-\underline{s})\} = \frac{1}{(2\pi\sigma^2)^L} e^{-\frac{1}{2\sigma^2}|\underline{z}-\underline{s}|_2^2}, \tag{127}$$

with $|\underline{z}-\underline{s}|_2 = \sqrt{\sum_{l=1}^L |z_l - s_l|^2}$ the ℓ^2-norm of the complex vector $\underline{z}-\underline{s}$. That is, $\underline{\mathbf{z}} \sim \mathcal{CN}(\underline{s}, 2\sigma^2 I_L)$. Note that Eq. (127) reduces to Eq. (76) if $L = 1$.

4.2. Statistical distribution of multiple-coil magnitude images

When images are acquired with multiple receiver coils and the *k*-space is fully sampled, a composite magnitude image can be obtained by pixelwise taking the root of the sum of squares (SoS) [61]:

$$\mathbf{m}_L = \sqrt{\sum_{l=1}^L \left(\mathbf{z}_{\mathrm{R}_l}^2 + \mathbf{z}_{\mathrm{I}_l}^2\right)}, \tag{128}$$

with L the number of coils and $\mathbf{z}_{\mathrm{R}_l}$ and $\mathbf{z}_{\mathrm{I}_l}$ the real and imaginary component of the complex image obtained from the raw data acquired by the lth coil. Note that we again suppose that all the equations are pixelwise and write $\mathbf{m}_L$ instead of $\mathbf{m}_L(\underline{r})$. If the variance of the noise at each coil is the same and there are no correlations, the PDF of $\mathbf{m}_L$ is given by Ref. [62,63]

$$f_{\mathbf{m}_L}(m) = \frac{m^L}{\sigma^2 a^{L-1}} e^{-\frac{m^2+a^2}{2\sigma^2}} I_{L-1}\left(\frac{ma}{\sigma^2}\right)\varepsilon(m), \tag{129}$$

where $a^2 = \underline{s}^H\underline{s} = \sum_{l=1}^L (s_{\mathrm{R}_l}^2 + s_{\mathrm{I}_l}^2) = \sum_{l=1}^L \left|s_l\right|^2$, with s_{R_l} and s_{I_l} the means of the real and imaginary components of the complex image pixel values obtained from raw data acquired with the lth coil. The PDF (129) is known as the *generalized Rice distribution* and is directly related to the so-called non-central chi $(nc-\chi)$ distribution. Indeed, it can be shown that the scaled random variable $\mathbf{m}_{L'} = \mathbf{m}_L/\sigma$, being the root sum of squares of a set of $2L$ independent Gaussian random variables with unit variance, has a $nc-\chi$ distribution, with $2L$ degrees of freedom and non-centrality parameter

$$\lambda = \sqrt{\sum_{l=1}^L \left(\left(\frac{s_{\mathrm{R}_l}}{\sigma}\right)^2 + \left(\frac{s_{\mathrm{I}_l}}{\sigma}\right)^2\right)} = \frac{a}{\sigma}. \tag{130}$$

Its PDF is given by

$$f_{\mathbf{m}_{L'}}(m) = \frac{m^L e^{-\frac{m^2+\frac{a^2}{\sigma^2}}{2}}}{\left(\frac{a}{\sigma}\right)^{L-1}} I_{L-1}\left(\frac{ma}{\sigma}\right)\varepsilon(m). \tag{131}$$

It follows from Eqs. (129) and (131), that $f_{\mathbf{m}_L}(m) = (1/\sigma) f_{\mathbf{m}_{L'}}(m/\sigma)$. This relation can also be derived directly using basic theory on random variable transformation [64]. In fact, both Eqs. (129) and (131) are often referred to as nc-χ distributions. In the remainder of this paper, we will follow this convention and refer to Eq. (129) as a $nc-\chi$ distribution as well.

When $a \to 0$, the PDF of $\mathbf{m}_L$ turns into a generalized Rayleigh PDF [31,62]:

$$f_{\mathbf{m}_L}(m) = \frac{2m^{2L-1}}{\left(\sigma\sqrt{2}\right)^{2L}\Gamma(L)} \exp\left(-\frac{m^2}{2\sigma^2}\right)\varepsilon(m), \tag{132}$$

which can be rewritten as [65]

$$f_{\mathbf{m}_L}(m) = \frac{1}{\Gamma(L)\sigma^2}\left(\frac{m}{2\sigma^2}\right)^{L-1} m^L \exp\left(-\frac{m^2}{2\sigma^2}\right)\varepsilon(m). \tag{133}$$

The moments of the generalized Rice PDF can be expressed analytically as [66]:

$$\mathbb{E}[\mathbf{m}_L^r] = \left(2\sigma^2\right)^{r/2} \frac{\Gamma[(2L+r)/2]}{\Gamma(L)}\, {}_1F_1\left(-\frac{r}{2}, L; 3 - \frac{a^2}{2\sigma^2}\right). \tag{134}$$

or, equivalently [67],

$$\mathbb{E}[\mathbf{m}_L^r] = \left(2\sigma^2\right)^{r/2} e^{-\frac{m^2+a^2}{2\sigma^2}} \frac{\Gamma[(2L+r)/2]}{\Gamma(L)}\, {}_1F_1\left(\frac{2L+r}{2}, L; \frac{a^2}{2\sigma^2}\right), \tag{135}$$

using the transformation ${}_1F_1(a,b,z) = e^z\, {}_1F_1(b-a, b, -z)$ [45]. The mean of $\mathbf{m}_L$ is given by Ref. [31]

$$\mathbb{E}[\mathbf{m}_L] = \sqrt{2}\sigma \frac{\Gamma\left(L+\frac{1}{2}\right)}{\Gamma(L)}\, {}_1F_1\left(-\frac{1}{2}, L; -\frac{a^2}{2\sigma^2}\right). \tag{136}$$

Again, the even moments turn out to be simple polynomials:

$$\mathbb{E}\left[\mathbf{m}_L^2\right] = 2L\sigma^2 + a^2, \tag{137}$$

$$\mathbb{E}\left[\mathbf{m}_L^4\right] = 4L^2\sigma^4 + 4L\sigma^4 + 4a^2L\sigma^2 + 4a^2\sigma^2 + a^4. \tag{138}$$

For $a = 0$, we obtain the moments of the generalized Rayleigh PDF:

$$\mathbb{E}[\mathbf{m}_L^r] = \left(2\sigma^2\right)^{r/2} \frac{\Gamma[(2L+r)/2]}{\Gamma(L)}, \tag{139}$$

which, using some general properties of the Gamma function $\Gamma(\cdot)$, yields for the mean value and variance [65]

$$\mathbb{E}[\mathbf{m}_L] = \frac{1\cdot 3\cdot 5\cdots(2L-1)}{2^{L-1}(L-1)!}\sqrt{\frac{\pi}{2}}\sigma, \tag{140}$$

and

$$\begin{aligned} Var(\mathbf{m}_L) &= \mathbb{E}\left[\mathbf{m}_L^2\right] - \mathbb{E}[\mathbf{m}_L]^2 \\ &= \left(2L - \left(\frac{1\cdot 3\cdot 5\cdots(2L-1)}{2^{L-1}(L-1)!}\right)^2 \frac{\pi}{2}\right)\sigma^2. \end{aligned} \tag{141}$$

Furthermore, it can be shown that the PDF of the random variable

$$\mathbf{q}_L = \mathbf{m}_L^2 = \sum_{l=1}^L \left(\mathbf{z}_{\mathrm{R}_l}^2 + \mathbf{z}_{\mathrm{I}_l}^2\right) \tag{142}$$

is given by Ref. [31]

$$f_{\mathbf{q}_L}(q) = \frac{1}{2\sigma^2} e^{-\frac{a^2+q}{2\sigma^2}} \left(\frac{q}{a^2}\right)^{\frac{L-1}{2}} I_{L-1}\left(\frac{\sqrt{q}a}{\sigma^2}\right)\varepsilon(q). \tag{143}$$

The PDF (143) is directly related to the so-called non-central chi-square PDF. Indeed, it can be shown that the random variable $\mathbf{q}_{L'} = \mathbf{q}_L/\sigma^2$, being the sum of squares of a set of $2L$ independent Gaussian random variables with unit variance, has a non-central chi-square $(nc-\chi^2)$ distribution, with $2L$ degrees of freedom and non-centrality parameter a^2/σ^2. Its PDF is given by Ref. [68]

$$f_{\mathbf{q}_L'}(q) = \frac{1}{2} e^{-\frac{\frac{a^2}{\sigma^2}+q}{2}} \left(\frac{q\sigma^2}{a^2}\right)^{\frac{L-1}{2}} I_{L-1}\left(\sqrt{q}\ \frac{a}{\sigma}\right)\varepsilon(q). \tag{144}$$

It follows from Eqs. (143) and (144), that $f_{\mathbf{q}_L}(q) = (1/\sigma^2) f_{\mathbf{q}_L'}(q/\sigma^2)$. This relation can also be derived directly using basic theory on random variable transformation [64]. In fact, since Eq. (143) is often also referred to as a $nc-\chi^2$ distribution [57,60,69], we will from now on refer to both Eqs. (143) and (144) as $nc-\chi^2$ distributions. The variance of $\mathbf{m}_L^2$ follows directly from Eqs. (137) and (138):

$$Var\left(\mathbf{m}_L^2\right) = \mathbb{E}\left[\mathbf{m}^4\right] - \left(\mathbb{E}\left[\mathbf{m}^2\right]\right)^2 = 4a^2\sigma^2 + 4L\sigma^4. \tag{145}$$

It is worthwhile mentioning that the generalized Rice distribution also applies to MR data acquired in Phase Contrast Magnetic Resonance (PCMR) imaging, which is a technique that is widely used to detect flow [31,32,70].

Recall that to arrive at the generalized Rice distribution (129) and the $nc-\chi^2$ distribution (143), we had to assume that there are no correlations between the coils and the variance of the noise at each coil is the same. In mathematical terms, these assumptions imply that the conditions (124) and (125) should be satisfied. However, in phased array (multiple-coil) systems noise correlations may exist [54,61,71]. Furthermore, the noise variance may differ from coil to coil. Generally, the noise correlation matrix could be determined experimentally from a reasonably large set of samples reflecting mere noise [60,72]. Taking noise correlations into account, Aja-Fernández et al. [57,60] considered the case in which $\Sigma_R = \Sigma_I$ is allowed to have nonzero off-diagonal elements, where the off-diagonal elements represent the correlations between each pair of coils. Holding on assumption (124), this yields $\underline{\mathbf{z}} \sim \mathcal{CN}(\mathbb{E}[\underline{\mathbf{z}}], \Sigma_z)$, with $\Sigma_z = 2\Sigma_R$. In this more general case, the PDF of $\mathbf{m}_L^2$ cannot be derived, but the mean and variance are given by Ref. [19,57]

$$\mathbb{E}\left[\mathbf{m}_L^2\right] = a^2 + 2tr(\Sigma_R), \tag{146}$$

$$Var\left(\mathbf{m}_L^2\right) = 4\underline{s}^H \Sigma_R \underline{s} + 4\|\Sigma_R\|_F^2 = 4\underline{s}^H \Sigma_R \underline{s} + 4tr(\Sigma_R), \tag{147}$$

with $\|\cdot\|_F$ the Frobenius norm and $tr(\bullet)$ the trace operator. Note that the mean (146) remains unaffected by noise correlation.

Aja-Fernández et al. [60] show that although the data in this case is not strictly $nc-\chi^2$ distributed, in practical cases this distribution is still a very accurate approximation if so-called *effective* parameters are considered. By using the method of moments, the so-called effective number of coils L_{eff} and the effective noise variance σ_{eff}^2 can be derived [60]:

$$L_{\text{eff}} = \frac{a^2 tr(\Sigma_R) + (tr(\Sigma_R))^2}{\underline{s}^H \Sigma_R \underline{s} + \|\Sigma_R\|_F^2}, \tag{148}$$

$$\sigma_{\text{eff}}^2 = \frac{tr(\Sigma_R)}{L_{\text{eff}}}. \tag{149}$$

Generally, noise correlations will reduce the number of degrees of freedom of the $nc-\chi^2$ distribution and increase the effective variance of the noise. Note that L_{eff} and σ_{eff}^2 depend on the signal and hence on the position within the image. As a result, the statistics of the noise will be spatially variant, and the noise becomes non-stationary [73].

Parameter estimation

Now that we have analyzed the statistical distribution(s) of MR images, we will next show how knowledge of this distribution can be used to estimate parameters from these images with optimal accuracy and precision. Furthermore, we will address the question as to what precision may be achieved ultimately from a particular MR image.

Suppose that one wants to estimate a parameter vector $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_K)^T$ from a set of N data points (i.e., observations) $\mathbf{w}_1,\ldots,\mathbf{w}_N$ that have a joint PDF

$$f_{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_N}\left(w_1, w_2, \ldots, w_N; \underline{\theta}\right), \tag{150}$$

which depends on $\underline{\theta}$. The observations may represent pixel values of an MR complex, magnitude or phase image, whereas the parameters $\underline{\theta}$ may, for example, represent the underlying true amplitude and phase values [2,46], or proton densities and relaxation times [5]. In Section 5.1, it will be shown how the parameterized PDF (150) can be used to compute the so-called Cramér-Rao lower bound (CRLB), which is a lower bound on the variance of any unbiased estimator of the parameters. Then, in Section 5.2, it will be shown how from the same PDF the maximum likelihood (ML) estimator, having favorable statistical properties, may be derived.

The Cramér-Rao lower bound

Obviously, different estimators can be used to estimate $\underline{\theta}$. To assess and compare their performances, quality measures such as accuracy and precision can be used. The accuracy of an estimator is expressed in terms of its bias, which is defined as the deviation of the expected value of the estimator from the true value of the parameter:

$$b\left(\widehat{\underline{\theta}}\right) = \mathbb{E}\left[\widehat{\underline{\theta}}\right] - \underline{\theta}. \tag{151}$$

The bias represents the systematic error. If the bias of an estimator is zero, the estimator is called unbiased. The precision of an estimator is defined by its variance, or, in the more general case of vector valued parameters, by its covariance matrix:

$$\text{cov}\left(\widehat{\underline{\theta}}\right) = \mathbb{E}\left[\left(\widehat{\underline{\theta}} - \mathbb{E}\left[\widehat{\underline{\theta}}\right]\right)\left(\widehat{\underline{\theta}} - \mathbb{E}\left[\widehat{\underline{\theta}}\right]\right)^T\right]. \tag{152}$$

The diagonal elements of $\text{cov}(\widehat{\underline{\theta}})$ represent the variances of the elements of $\widehat{\underline{\theta}}$, whereas the non-diagonal elements represent the covariances between the elements of the estimator. Note, that precision is a measure of the non-systematic error. It concerns the spread of the estimates if the experiment is repeated under the same conditions. Generally, different estimators will have different precisions. However, it can be shown that under general regularity conditions the covariance matrix of any unbiased estimator $\widehat{\underline{\theta}}$ satisfies [74]

$$\text{cov}\left(\widehat{\underline{\theta}}\right) \geq F^{-1}, \tag{153}$$

with

$$F = -\mathbb{E}\left[\frac{\partial^2 \ln f_{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_N}\left(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N; \underline{\theta}\right)}{\partial\underline{\theta}\partial\underline{\theta}^T}\right] \tag{154}$$

the so-called called *Fisher information matrix*. Inequality (153) expresses that the difference between the left-hand and right-hand member is positive semi-definite. A property of a positive semi-definite matrix is that its diagonal elements cannot be negative. This means that the diagonal elements of $\mathrm{cov}(\hat{\underline{\theta}})$, that is, the variances of the elements of $\hat{\underline{\theta}}$, are larger than or equal to the corresponding diagonal elements of F^{-1}. Hence, F^{-1} represents a lower bound to the variances of all unbiased $\hat{\underline{\theta}}$. The matrix F^{-1} is called the *Cramér-Rao Lower Bound (CRLB)*.

Maximum likelihood estimation

To construct the maximum likelihood (ML) estimator of the unknown parameter $\underline{\theta}$ from a set of available observations $w_1,\ldots,w_N$, we substitute these observations (i.e., numbers) for the corresponding independent variables in Eq. (150). The expression that results depends only on the unknown parameters $\underline{\theta}$. If we now regard these parameters as variables, the deterministic function

$$L\left(\underline{\theta}; w_1, w_2, \ldots, w_N\right) \tag{155}$$

that results is called the *likelihood function*. The Maximum Likelihood estimate $\hat{\underline{\theta}}_{\mathrm{ML}}$ of the parameter $\underline{\theta}$ is now defined as the value of $\underline{\theta}$ that maximizes the likelihood function:

$$\left\{\hat{\underline{\theta}}_{\mathrm{ML}}\right\} = \arg\max_{\underline{\theta}} L\left(\underline{\theta}; w_1, w_2, \ldots, w_N\right) \tag{156}$$

or, equivalently,

$$\left\{\hat{\underline{\theta}}_{\mathrm{ML}}\right\} = \arg\max_{\underline{\theta}} \ln L\left(\underline{\theta}; w_1, w_2, \ldots, w_N\right) \tag{157}$$

in which $\ln L(\cdot)$ is called the *log-likelihood function*. The ML estimator has a number of favorable statistical properties [75]. First, it can be shown that this estimator achieves the CRLB asymptotically, that is, for an infinite number of observations. Therefore, it is asymptotically most precise (or, *asymptotically efficient*). Second, it can be shown that the ML estimator is consistent, which means that it converges to the true value of the parameter in a statistically well defined way if the number of observations increases. Third, the ML estimator is asymptotically normally distributed, with a mean equal to the true value of the parameter and a covariance matrix equal to the CRLB. If these asymptotic properties also apply to a finite or even small number of observations can often only be assessed by estimating from artificial, simulated observations. Finally, the ML estimator is known to have the invariance property. That is, if $\hat{\underline{\theta}}_{ML}$ is the ML estimator of $\underline{\theta}$, and if $g(\underline{\theta})$ is any function of $\underline{\theta}$, then the ML estimator of $\alpha = g(\underline{\theta})$ is given by $\hat{\alpha} = g(\hat{\underline{\theta}}_{ML})$.

Note that the observations (i.e., pixels) $m_1,\ldots,m_N$ from which the parameter of interest $\underline{\theta}$ is estimated (using the ML estimator) can be selected locally or non-locally. In the first case, the parameter is estimated from pixels in a local neighborhood within which the parameter is assumed to be constant (cfr., [46,66]). In the second case, the pixels for the ML estimation of the true underlying parameter are selected in a non local way based on, for example, the intensity similarity of the pixel neighborhoods. This similarity can be measured using, for example, the Euclidean distance [76,77], sparseness in a transform domain [78,79], or, as proposed recently, the Kolmogorov–Smirnov (KS) test [80].

Estimation of signal and noise from magnitude images

To illustrate the practical application of the theory summarized in Section 5, we will now consider, as an illustrative example, the problem of estimating the underlying true signal amplitude from a single-coil magnitude image. For this estimation problem, the CRLB and the ML estimator are derived. For the case of multiple-coil images, the CRLB and ML estimator can be derived in a similar way (see, e.g., [67,81]).

Consider a set of N independent pixel values of a magnitude image taken from a region in which the underlying true signal amplitude a is assumed to be constant. The joint PDF $f_{\mathbf{m}_1,\mathbf{m}_2,\ldots,\mathbf{m}_N}(m_1,m_2,\ldots,m_N)$ of these pixel values (from now on called *observations*) is then given by the product of the marginal PDFs $f_{\mathbf{m}_i}(m_i)$ of the individual observations constituting this set:

$$f_{\mathbf{m}_1,\mathbf{m}_2,\ldots,\mathbf{m}_N}\left(m_1, m_2, \ldots, m_N; a, \sigma^2\right) = \prod_{i=1}^{N} f_{\mathbf{m}_i}\left(m_i; a, \sigma^2\right). \tag{158}$$

For single-coil images, $f_{\mathbf{m}_i}(m_i; a, \sigma^2)$ is given by the Rician PDF Eq. (78). Note that the joint PDF depends on the true signal amplitude a and the noise standard deviation σ, as expressed in the notation used.

6.1. CRLB

The CRLB can be derived from Eqs. (153)–(154) and (158) and (78). If the noise variance σ^2 is known, the CRLB, is given by Ref. [9]:

$$\mathrm{CRLB} = \frac{\sigma^2}{N}\left(\eta - \frac{a^2}{\sigma^2}\right)^{-1}, \tag{159}$$

with

$$\eta = \mathbb{E}\left[\frac{\mathbf{m}^2}{\sigma^2}\frac{I_1^2\left(\frac{a\mathbf{m}}{\sigma^2}\right)}{I_0^2\left(\frac{a\mathbf{m}}{\sigma^2}\right)}\right], \tag{160}$$

The expectation value in Eq. (160) can be evaluated numerically. If the noise variance σ^2 is unknown and has to be estimated simultaneously with the signal parameter a, the elements of the Fisher information matrix F with respect to the parameter vector $\underline{\theta} = (a, \sigma^2)^T$ are given by Ref. [2]:

$$F(1,1) = \frac{N}{\sigma^2}\left(\eta - \frac{a^2}{\sigma^2}\right) \tag{161}$$

$$F(1,2) = F(2,1) = \frac{Na}{\sigma^4}\left(1 + \frac{a^2}{\sigma^2} - \eta\right) \tag{162}$$

$$F(2,2) = \frac{N}{\sigma^4}\left(1 + \frac{a^2}{\sigma^2}(\eta - 1) - \frac{a^4}{\sigma^4}\right) \tag{163}$$

where $F(i,j)$ denotes the (i,j)th element of the matrix F and η is given by Eq. (160). Finally, the CRLB for unbiased estimation of (a,σ^2) is obtained by simple inversion of the 2×2 matrix F.

Maximum likelihood estimator

To construct the Maximum Likelihood (ML) estimator of the unknown parameters a and σ^2 from a set of available magnitude observations $m_1,\ldots,m_N$ we substitute these observations (i.e., numbers) for the corresponding independent variables in Eq. (158). The thus obtained Likelihood function is then given by

$$L\left(a, \sigma^2; m_1, m_2, \ldots, m_N\right) = \prod_{i=1}^{N} \frac{m_i}{\sigma^2} e^{-\frac{a^2+m_i^2}{2\sigma^2}} I_0\left(\frac{m_i a}{\sigma^2}\right) \tag{164}$$

and the Maximum Likelihood estimates $\widehat{a}_{ML}$ and $\widehat{\sigma}_{ML}$ of the parameters a and σ are found by maximizing the likelihood function with respect to a and σ [2]:

$$\left\{\widehat{a}_{ML}, \widehat{\sigma}^2{}_{ML}\right\} = \arg\max_{a,\sigma^2} \prod_{i=1}^{N} \frac{m_i}{\sigma^2} e^{-\frac{a^2+m_i^2}{2\sigma^2}} I_0\left(\frac{m_i a}{\sigma^2}\right), \tag{165}$$

or, equivalently, since the logarithm is a monotonically increasing function,

$$= \arg\max_{a,\sigma^2} \ln L\left(a, \sigma^2; m_1, m_2, \ldots, m_N\right)$$

$$\{\widehat{a}_{ML}, \widehat{\sigma}^2{}_{ML}\} = \arg\max_{a,\sigma^2} \ln \prod_{i=1}^{N} \frac{m_i}{\sigma^2} e^{-\frac{a^2+m_i^2}{2\sigma^2}} I_0\left(\frac{m_i a}{\sigma^2}\right)$$

$$= \arg\max_{a,\sigma^2} \left(\sum_{i=1}^{N} \ln\left(\frac{m_i}{\sigma^2}\right) - \sum_{i=1}^{N} \frac{m_i^2 + a^2}{2\sigma^2} + \sum_{i=1}^{N} \ln I_0\left(\frac{a m_i}{\sigma^2}\right)\right). \tag{166}$$

Note that if the noise variance σ^2 is known, Eq. (166) simplifies to

$$\widehat{a}_{ML} = \arg\max_{a} \left(\sum_{i=1}^{N} \ln I_0\left(\frac{a m_i}{\sigma^2}\right) - \frac{N a^2}{2\sigma^2}\right). \tag{167}$$

Yakovleva et al. [82] recently introduced a new technique to calculate the ML estimates of a and σ^2. This technique effectively reduces the task of solving a system of two nonlinear equations with two unknown variables, to the task of solving just one equation with one unknown variable. Using Yakovleva's technique, finding the ML estimates of both σ^2 and a is therefore not more complicated (in terms of computational cost) than finding the ML estimate of a only (with σ^2 known).

Discussion

Note that in the more general case, the underlying signal amplitude a can be a parametric function $f(\underline{\theta})$ of an unknown parameter vector $\underline{\theta}$, where typical elements of $\underline{\theta}$ are proton density ρ and relaxation time constants T_1, T_2, T_2^*. In this case, the same theory (described in Section 5) that was used to derive the CRLB and ML estimator of the parameter a, can straightforwardly be applied to derive the CRLB and ML estimator of $\underline{\theta}$ [5].

Furthermore, ML estimates can also be obtained from the complex valued images. It was shown by Sijbers and den Dekker [2] that ML estimation of the underlying true signal amplitude from complex data with equal underlying true phase values is generally better in terms of the mean squared error (MSE) than ML estimation from magnitude data. However, if the phase values vary within the region from which the amplitude is estimated, ML estimation from magnitude data is significantly better in terms of the MSE.

Finally, for a list of series expansions, recursive relations and polynomial approximations of modified Bessel functions that have been proven useful in the numerical calculation of ML estimates from magnitude images, the reader is referred to Appendix C.

Discussion and conclusions

In this review paper, it has been shown that the raw complex MR data points acquired in the spatial frequency domain (i.e., the k-space) are characterized by a joint circularly complex Gaussian distribution, with a diagonal covariance matrix. After taking the inverse DFT, we obtain a complex image in the spatial domain (i.e., the image space). Due to the linearity and the orthogonality of the DFT, the pixels of this so-called reconstructed image are also jointly circularly complex Gaussian distributed with a diagonal covariance matrix. Taking the magnitude and phase, however, are nonlinear operations. Therefore, magnitude and phase images are no longer Gaussian distributed. The PDFs of magnitude and phase images have been described in this paper. In particular, it has been shown that the pixels of magnitude images obtained by single-coil acquisition are Rician distributed, whereas magnitude images acquired using a multiple-coil system (and using the sum of squares reconstruction algorithm) are $nc-\chi$ distributed, under the assumption that the noise variance at each coil is the same and there are no inter-coil noise correlations. If this assumption is not valid, the noise in the magnitude images becomes spatially non-stationary. In this case, it is no longer possible to derive an exact expression for the distribution of the image pixel values, although, under certain conditions, accurate approximations may still be given.

Furthermore, it has been summarized how knowledge of the distribution of MR images can be used to (i) derive the precision that may be achieved ultimately when estimating parameters from a particular MR image and (ii) to construct the maximum likelihood (ML) estimator, which achieves this precision at least asymptotically.

Finally, we note that data distributions in MR images that were generated using nonlinear reconstruction techniques may be very different from those of conventional Fourier based reconstruction [83–86]. Nonlinear reconstruction techniques have been shown to be successful in reconstructing high-resolution images from subsampled data. Such techniques are becoming increasingly popular, as the demand for shorter scan times without significantly affecting image quality increases. It has been noted [87] that although the convergence and other deterministic properties of nonlinear reconstruction methods are well established, little is known about how noise in the source data influences noise in the final reconstructed image. In Refs. [87], the noise distribution from nonlinear reconstructed MR images was determined in an experimental way. Depending on the level of subsampling, the noise distribution was observed to vary from a Rayleigh distribution to a log-normal distribution with increasing level of subsampling. For future research, it would be highly valuable to fully characterize the MR data distribution from solid theoretical derivations.

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Appendix A. The complex multivariate Gaussian distribution

The following analysis is to a large extent based on Ref. [59]. Let $\underline{\mathbf{z}}_R$ and $\underline{\mathbf{z}}_I$ be the real vectors

$$\underline{\mathbf{z}}_R = \left(\underline{\mathbf{z}}_{R_1} \cdots \underline{\mathbf{z}}_{R_L}\right)^T \tag{A.1}$$

and

$$\underline{\mathbf{z}}_I = \left(\underline{\mathbf{z}}_{I_1} \cdots \underline{\mathbf{z}}_{I_L}\right)^T \tag{A.2}$$

with jointly Gaussian distributed random variables and define

$$\underline{\boldsymbol{t}} = \begin{pmatrix} \underline{\mathbf{z}}_R \\ \underline{\mathbf{z}}_I \end{pmatrix} \tag{A.3}$$

and

$$\underline{\mathbf{z}} = (\mathbf{z}_1 \ldots \mathbf{z}_L)^T, \tag{A.4}$$

with $\mathbf{z}_l = \mathbf{z}_{R_l} + \imath \mathbf{z}_{I_l}$. Next, define

$$\underline{\boldsymbol{w}} = \begin{pmatrix} \underline{\mathbf{z}} \\ \underline{\mathbf{z}}^* \end{pmatrix}. \tag{A.5}$$

The covariance matrix of the complex vector $\underline{\mathbf{z}}$ is defined by its (m,n)th element

$$\text{cov}(\mathbf{z}_m, \mathbf{z}_n) = \mathbb{E}\left[(\mathbf{z}_m - \mathbb{E}[\mathbf{z}_m])(\mathbf{z}_n^* - \mathbb{E}[\mathbf{z}_n^*])\right]. \tag{A.6}$$

Therefore,

$$\text{cov}\left(\underline{\mathbf{z}}, \underline{\mathbf{z}}\right) = \mathbb{E}\left[\left(\underline{\mathbf{z}} - \mathbb{E}\left[\underline{\mathbf{z}}\right]\right)\left(\underline{\mathbf{z}} - \mathbb{E}\left[\underline{\mathbf{z}}\right]\right)^H\right]. \tag{A.7}$$

Similarly,

$$\text{cov}\left(\underline{\mathbf{w}}, \underline{\mathbf{w}}\right) = \begin{pmatrix} \text{cov}\left(\underline{\mathbf{z}}, \underline{\mathbf{z}}\right) & \text{cov}\left(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*\right) \\ \text{cov}\left(\underline{\mathbf{z}}^*, \underline{\mathbf{z}}\right) & \text{cov}\left(\underline{\mathbf{z}}^*, \underline{\mathbf{z}}^*\right) \end{pmatrix}, \tag{A.8}$$

with $\text{cov}(\underline{\mathbf{z}}^*, \underline{\mathbf{z}}) = \text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*)^H$, where $\text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*)$ is known as the *pseudo-covariance* matrix of $\underline{\mathbf{z}}$ [88]. Furthermore, it can be shown that $\text{cov}(\underline{\mathbf{z}},\underline{\mathbf{z}})$ and $\text{cov}(\underline{\mathbf{z}},\underline{\mathbf{z}}^*)$ can be expressed in terms of the covariance matrices of $\underline{\mathbf{z}}_R$ and $\underline{\mathbf{z}}_I$:

$$\text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*) = \Sigma_R - \Sigma_I + \imath(\Sigma_{IR} + \Sigma_{RI}), \tag{A.9}$$

$$\text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}) = \Sigma_R + \Sigma_I + \imath(\Sigma_{IR} - \Sigma_{RI}). \tag{A.10}$$

with

$$\Sigma_R = \text{cov}(\underline{\mathbf{z}}_R, \underline{\mathbf{z}}_R), \tag{A.11}$$

$$\Sigma_I = \text{cov}(\underline{\mathbf{z}}_I, \underline{\mathbf{z}}_I), \tag{A.12}$$

and

$$\Sigma_{IR} = \Sigma_{RI}^T = \text{cov}(\underline{\mathbf{z}}_I, \underline{\mathbf{z}}_R). \tag{A.13}$$

Since the elements of $\underline{\boldsymbol{t}}$ are jointly Gaussian distributed, it can be shown that the PDF of $\underline{\mathbf{z}}$ is given by Ref. [59]:

$$\frac{1}{\pi^L \sqrt{\det(\Sigma_w)}} \exp\left\{ -\frac{1}{2}\left(\underline{w} - \mathbb{E}\left[\underline{\mathbf{w}}\right]\right)^H \Sigma_w^{-1} \left(\underline{w} - \mathbb{E}\left[\underline{\mathbf{w}}\right]\right)\right\} \tag{A.14}$$

with $\Sigma_w = \text{cov}(\underline{\mathbf{w}}, \underline{\mathbf{w}})$. This is usually denoted as $\underline{\mathbf{z}} \sim \mathcal{CN}(\mathbb{E}[\underline{\mathbf{z}}], \Sigma_z, C)$, with $\Sigma_z = \text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}})$ the covariance matrix of $\underline{\mathbf{z}}$ and $C = \text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*)$ the pseudo-covariance matrix of $\underline{\mathbf{z}}$.

Next, consider the special case that

$$\mathbb{E}[(\mathbf{z}_m - \mathbb{E}[\mathbf{z}_m])(\mathbf{z}_n - \mathbb{E}[\mathbf{z}_n])] = \mathbb{E}[(\mathbf{z}_m - \mathbb{E}[\mathbf{z}_m])^*(\mathbf{z}_n - \mathbb{E}[\mathbf{z}_n])^*] = 0, \tag{A.15}$$

that is, $\text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*)$ and $\text{cov}(\underline{\mathbf{z}}^*, \underline{\mathbf{z}})$ are $L \times L$ null matrices. Then, the complex random variables $\underline{\mathbf{z}}$ are called *circularly complex* and

$$\Sigma_w = \begin{pmatrix} \Sigma_z & O \\ O & \Sigma_z^* \end{pmatrix}, \tag{A.16}$$

where O is the $L \times L$ null matrix. Substituting Eq. (A.16) in (A.14) yields the so-called joint circularly complex normal distribution [59], also known as the complex multivariate normal (or Gaussian) PDF [19,27]:

$$f_{\underline{\mathbf{z}}}\left(\underline{z}\right) = \frac{1}{\pi^L \det(\Sigma_z)} \exp\left\{ -\left(\underline{z} - \mathbb{E}\left[\underline{\mathbf{z}}\right]\right)^H \Sigma_z^{-1} \left(\underline{z} - \mathbb{E}\left[\underline{\mathbf{z}}\right]\right)\right\}. \tag{A.17}$$

This is usually denoted as $\underline{\mathbf{z}} \sim \mathcal{CN}(\mathbb{E}[\underline{\mathbf{z}}], \Sigma_z)$. Note that complex random variables for which condition Eq. (A.15) holds (i.e., with a vanishing pseudo-covariance matrix) are often called *proper* [88]. Furthermore, note that it follows from Eqs. (A.9) and (A.10) that condition (A.15) is satisfied if and only if $\Sigma_R = \Sigma_I$ and $\Sigma_{IR} = -\Sigma_{RI} (= -\Sigma_{IR}^T)$, where the skew-symmetry of Σ_{IR} implies that Σ_{IR} has a zero main diagonal, which means that the real and imaginary part of each component $\mathbf{z}_k$ of $\underline{\mathbf{z}}$ are uncorrelated. Note that condition (A.15) also implies that for zero mean $\underline{\mathbf{z}}$ we have $\mathbb{E}[\mathbf{z}_k \mathbf{z}_l] = 0$. The vanishing of $\text{cov}(\underline{\mathbf{z}}, \underline{\mathbf{z}}^*)$ and $\text{cov}(\underline{\mathbf{z}}^*, \underline{\mathbf{z}})$ does not, however, imply that the real part of $\mathbf{z}_k$ and the imaginary part of $\mathbf{z}_l$ are uncorrelated for $k \neq l$ [88]. Finally, note that if condition (A.15) is satisfied, the covariance matrix Σ_z can be written as

$$\Sigma_z = 2\Sigma_R + 2\imath\Sigma_{IR}. \tag{A.18}$$

Appendix B. Covariance of two-dimensional DFT

Suppose that the 2D data set in the k-space consists of $M \times M$ complex data points. These data points can be described by a $M \times M$ matrix $\boldsymbol{Z}$, but the data can also be described by a vector of $N = M^2$ elements that is obtained by stacking the columns of $\boldsymbol{Z}$. Let's define this vector by

$$\underline{\mathbf{z}}\left(\underline{k}\right) = \begin{pmatrix} \mathbf{z}\left(\underline{k}_1\right) \\ \vdots \\ \mathbf{z}\left(\underline{k}_N\right) \end{pmatrix}, \tag{B.1}$$

with

$$\mathbb{E}\left[\underline{\mathbf{z}}\left(\underline{k}\right)\right] = \begin{pmatrix} s\left(\underline{k}_1\right) \\ \vdots \\ s\left(\underline{k}_N\right) \end{pmatrix}. \tag{B.2}$$

Similarly, the 2D data in the image space obtained by applying the 2D inverse discrete Fourier transform is described by the $N \times 1$ vector

$$\underline{\mathbf{z}}\left(\underline{r}\right) = \begin{pmatrix} \mathbf{z}\left(\underline{r}_1\right) \\ \vdots \\ \mathbf{z}\left(\underline{r}_N\right) \end{pmatrix}. \tag{B.3}$$

Now, it can be shown that

$$\underline{\mathbf{z}}\left(\underline{r}\right) = \frac{1}{N} X^H \underline{\mathbf{z}}\left(\underline{k}\right), \tag{B.4}$$

where X is an $N \times N$ matrix given by

$$X = A \otimes A, \tag{B.5}$$

where $\otimes$ denotes the Kronecker product and

$$A = \begin{pmatrix} \alpha^0 & \alpha^0 & \alpha^0 & \cdots & \alpha^0 \\ \alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{M-1} \\ \alpha^0 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^0 & \alpha^{(M-1)} & & \cdots & \alpha^{(M-1)^2} \end{pmatrix}, \tag{B.6}$$

with $\alpha = e^{-\imath 2\pi/M}$. Note that Eq. (B.4) is a linear operation. This implies that if $\underline{\mathbf{z}}(\underline{k})$ is Gaussian distributed, its inverse DFT $\underline{\mathbf{z}}(\underline{r})$ will also be Gaussian distributed. Furthermore, it can be shown straightforwardly that the covariance matrix of $\underline{\mathbf{z}}(\underline{r})$ is given by

$$\begin{aligned} \mathrm{Cov}\left(\underline{\mathbf{z}}\left(\underline{r}\right),\underline{\mathbf{z}}\left(\underline{r}\right)\right) &= \mathbb{E}\left[\left(\underline{\mathbf{z}}\left(\underline{r}\right) - \mathbb{E}\left[\underline{\mathbf{z}}\left(\underline{r}\right)\right]\right)\left(\underline{\mathbf{z}}\left(\underline{r}\right) - \mathbb{E}\left[\underline{\mathbf{z}}\left(\underline{r}\right)\right]\right)^H\right] \\ &= \frac{1}{N^2} X^H \Sigma_z X, \end{aligned} \tag{B.7}$$

with

$$\Sigma_z = \mathrm{cov}(\underline{\mathbf{z}}(\underline{k}), \underline{\mathbf{z}}(\underline{k})). \tag{B.8}$$

Note that if $\sum_z = 2\sigma_K^2 I_N$, with I_N the $N \times N$ diagonal matrix, expression (B.7) simplifies to

$$\mathrm{Cov}\left(\underline{\mathbf{z}}\left(\underline{r}\right),\underline{\mathbf{z}}\left(\underline{r}\right)\right) = \frac{2\sigma_K^2}{N^2} X^H X = \frac{2\sigma_K^2}{N} I_N, \tag{B.9}$$

since $X^H X = N I_N$. Finally, if

$$\underline{\mathbf{z}}\left(\underline{k}\right) \sim \mathcal{CN}\left(\underline{s},\ 2\sigma_K^2 I_N\right), \tag{B.10}$$

then

$$\underline{\mathbf{z}}\left(\underline{r}\right) \sim \mathcal{CN}\left(\frac{1}{N} X^H \underline{s}, 2\frac{\sigma_K^2}{N} I_N\right). \tag{B.11}$$

Appendix C. Modified Bessel functions of the first kind of integer order

The series expansion of the nth order modified Bessel function of the first kind is given by

$$I_{\mathbf{n}}(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{m!\Gamma(m+n+1)}. \tag{C.1}$$

Furthermore, the following recursive relationships hold:

$$I_{n+1}(x) = -\left(\frac{2n}{x}\right) I_n(x) + I_{n-1}(x) \tag{C.2}$$

$$I_{n-1}(x) + I_{n+1}(x) = 2I'_n(x), \tag{C.3}$$

with $I'_n(x) = d/dx I_n(x)$. Moreover,

$$I_{-n}(x) = I_n(x). \tag{C.4}$$

In the region $x \ll n$, $I_n(x)$ becomes, asymptotically, a simple power of its argument [89]

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \quad n \geq 0, \tag{C.5}$$

whereas in the region $x \gg n$, $I_n(x)$ is well approximated by Ref. [89]

$$I_n(x) \approx \frac{1}{\sqrt{2\pi x}} \exp(x). \tag{C.6}$$

For $|x| \leq 3.75$, the following polynomial approximations hold [45]

$$\begin{aligned} x^{\frac{1}{2}} e^{-x} I_0(x) = 0.39894228 &+ 0.01328592t^{-1} + 0.00225319t^{-2} \\ &- 0.00157565t^{-3} + 0.00916281t^{-4} \\ &- 0.02057706t^{-5} + 0.02635537t^{-6} \\ &- 0.01647633t^{-7} + 0.00392377t^{-8} + \varepsilon_1, \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} x^{\frac{1}{2}} e^{-x} I_1(x) = 0.39894228 &- 0.03988024t^{-1} - 0.00362018t^{-2} \\ &+ 0.00163801t^{-3} - 0.01031555t^{-4} \\ &+ 0.02282967t^{-5} - 0.02895312t^{-6} \\ &+ 0.01787654t^{-7} - 0.00420059t^{-8} + \varepsilon_2, \end{aligned} \tag{C.8}$$

with $t = x/3.75$, $|\varepsilon_1| < 1.9 \times 10^{-7}$ and $|\varepsilon_2| < 2.2 \times 10^{-7}$.

Practical algorithms for accurate numerical calculation of Bessel functions using the relations described above can be found in Ref. [89].

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