SIGNAL COMPRESSION USING COMPRESSED SENSING AND BEST K-TERM APPROXIMATION: AN OVERVIEW

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ABSTRACT **In this article, we introduce some of the most effective paradigms used in compressing signals having sparse representation in different domain. These methods are discussed here are compressed sensing (CS) and best k-term approximation (BkTA). In both manners, the signals are supposed to have sparsity in another domain (in this paper, just frequency domain is considered). By choosing an upper bound for the SNR loss, the sparsest level of supports might be estimated. Simulation and calculation of complexity point out the advantages as well as disadvantages of the two procedures, in which BkTA is going to be more efficient despite of being simpler. Consequently, this very short overview is to assert the existence of a compression paradigm which is perfectly used at the SNR loss at most 0.5 dB.**

KEY WORDS **Compressed Sensing (CS), Best k-Term Approximation (BkTA), Sparse Representation (SR), Discrete Fourier Transform (DFT), Discrete Wavelets Transform (DWT), Basis Pursuit (BP), Orthogonal Matching Pursuits (OMP), Tree-based Orthogonal Matching Pursuit (TOMP).**

1. INTRODUCTION

In telecommunication, a cognitive radio (CR) is a radio that might be dynamically programmed and configured to detect the best transmitting channels in its neighborhood to avoid user interference and congestion. More precisely, it is such a radio which is able to automatically access available channels in normal wideband spectrum, then consequently changes technical parameters to allow more concurrent wireless communications in a given spectrum band at one location. One of the main functions of cognitive radios is wideband spectrum sensing, which refers to spectrum sensing over large spectral bandwidth, typically hundreds of MHz or even several GHz. Traditional methodology requires avery large number of the computation since the standard ADC implementation is supposed to use the usual quantized Shannon representation[2]. Furthermore, it also cannot afford the high sampling rate with high resolution. As a result, revolutional techniques are required, e.g., sub-Nyquist sampling[7]. The theory of signal compression travels against the common sense in data acquisition, where one might recover a certain signal (speech, image, video,…) from a very far fewer measurements quantized from the ground truth.

By intuition, for signals having a sparsity one can "measure" them by recording just some components that have largest magnitudes together with their locations. Fortunately, most of signals in general having this property by transforming to a different domain, e.g., Fourrier transform. In detail, let $s(t)$ be a time-continuous signal represented by a vector $s \in \mathbb{R}^N$, its discrete Fourier transform (DFT) can be obtained by taking the matrix-vector multiplication

$$\hat{s} = F \bullet s, \qquad (1)$$

where F is the N-by-N DFT matrix

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}, \text{ with}$$

$\omega = e^{-2\pi i/N}$.

The sections below describe two of the most effective methods used in compressing these characteristic signals: Compressed sensing (CS) and best k-term approximation (BkTA).

2. COMPRESSED SENSING

2.1. PROBLEM FORMULATION

The novel problem is stated as followed: Given a time-continuous signal $s(t)$ which is sampled by the vector s so that the Shannon-Nyquist sampling theorem is satisfied, how one can locate it inside the memory such that i) the least memory capacity is used and ii) the ideal signal $s(t)$ can be recovered by some proposed algorithm which iii) does not prevent it from losing fundamental characteristics? Compressed sensing (CS) is a method which was born to resolve this issue. It is a signal processing technique for efficiently acquiring and reconstructing a signal, by finding solutions to underdetermined linear systems[4]. This is based on the principle that, through optimization, the sparsity of a signal can be exploited to recover it from far fewer samples than required by the Shannon-Nyquist sampling theorem[4][5][7].

The CS problem can be briefly formulated as followed: Given a vector $s \in \mathbb{R}^N$ be the signal to be compressed and $x \in \mathbb{R}^K$ where $K \ll N$ stands for the signal which "measures" the ideal signal s through the following linear transformation:

$$x = A \bullet s, \quad (2)$$

where $A \in \mathbb{R}^{K\times N}$ is a given matrix. Find a good approximation for the signal s.

The theory of linear algebra said that the equation (2) does not have unique solution. As a result, there is not any method to perfectly recover s from x. Nonetheless, if the original vector s has a sparsity, there might be a possibility to reconstruct s from x through some techniques in CS.

From (1) and (2), the following equality is obtained:

$$x = AF^{-1} \bullet \hat{s}, \quad (3)$$

where the vector is $\hat{s}$ is sparse. How can we estimate $\hat{s}$?

2.2. METHODOLOGIES

The most useful method to reconstruct $\hat{s}$ is to consider its smallest-Euclidean-length estimator, i.e., to minimize the mean square error in $\mathbb{R}^2$. In more particular, the so-called $\hat{s}$-recovery using l^2 is considered:

$$\hat{s}^* = argmin_{\hat{s}} \|\hat{s}\|_2, \quad (4)$$

subject to

$$x = AF^{-1} \bullet \hat{s}. \quad (5)$$

Here $\|\bullet\|_2$ denotes the Euclidean distance in the N-dimensional vector space $\mathbb{R}^N$, i.e, $\|v\|_2 = \sqrt{\sum_{i=1}^N v_i^2}$. However, this computation is often not in used since the solution $\hat{s}^*$ is basically incorrect compared to the ground truth. This can be briefly explained by the lack property of the sparsity: The solution of this least square problem might be seen as the shortest projection of the original onto a hyperplane in $\mathbb{R}^N$, which is basically not located on any axis! So, the signal obtained is not sparse.

In order to avoid this unexpected phenomena, the $\hat{s}$-recovery using l^0 is often in used:

$$\hat{s}^* = argmin_{\hat{s}} \|\hat{s}\|_0, \quad (6)$$

subject to

$$x = AF^{-1} \bullet \hat{s}. \quad (7)$$

Here$\|\bullet\|_0$ denotes the pseudo-norm which can be seen as the sparsity level in N-dimensional vector space, i.e, $\|v\|_0 = \# \{v_i \neq 0 \,|i = 1,2,\dots,N\}$. However, this is an NP-hard problem which requires a numerous complicated computations and cannot be solved in a polynomial time. Hence, in term of scientific computing, l^1 is more comfortable in used:

$$\hat{s}^* = argmin_{\hat{s}} \|\hat{s}\|_1, \quad (8)$$

subject to

$$x = AF^{-1} \bullet \hat{s}. \quad (9)$$

Here $\|\bullet\|_1$ denotes the l^1-norm in $\mathbb{R}^N$, i.e, $\|v\|_1 = \sum_{i=1}^N |v_i|$.

This is so-called Basis Pursuit (BP).

2.3. ALGORITHMS

This section introduces some reconstruction methods related to $\hat{s}$-recovery problems using l^1 and l^0, respectively. The sensing matrix in each part should satisfies the restricted isometry property (RIP) condition and the incoherent sampling condition[5][7]. In particular, it is shown that with exponentially high probability, random Gaussian[5], Bernoulli[10], and partial Fourier matrices satisfy the RIP with number of measurements nearly linear in the sparsity level.

BASIS PURSUIT (BP)

In fact, the BP is not a linear programming problem (LPP). However, by a simple mathematical transformation, it becomes a LPP. More precisely, by adding new variables $t_1, t_2, t_3, \dots, t_N$ so that

$$|\hat{s}_i| \leq t_i \quad (10)$$

for all $1 \leq i \leq N$, the problem becomes a LPP which might be easily solved by Danzig's simplex method:

$$\hat{s}^* = argmin_{\hat{s}} \langle e, t \rangle, \quad (11)$$

subject to

$$I\hat{s} - It \ \leq 0, \quad (12)$$

$$I\hat{s} + It \ \geq 0, \quad (13)$$

$$AF^{-1}\hat{s} = x, \quad (14)$$

where I is the identity matrix of order N and $e = [1, 1, 1, \dots, 1, 1]^T \in \mathbb{R}^N$. The $\hat{s}$-recovery using l^1 is practicallyspecificized from the following more general problem:

$$\hat{s}^* = argmin_{\hat{s}} \|\hat{s}\|_1, \quad (15)$$

subject to

$$\|AF^{-1} \bullet \hat{s} - x\|_2 < \delta, \quad (16)$$

Where δ is a sufficiently small.

It is proved that this issue is equivalent to the Basis Pursuit Denoising (BPD) problem which is stated as followed:

$$\hat{s}^* = argmin_{\hat{s}} \left(\frac{1}{2} \|AF^{-1} \bullet \hat{s} - x\|_2 + \lambda \|\hat{s}\|_1 \right). \quad (17)$$

In both cases, it is obvious that the problems are equivalent to the BP when $\lambda \to 0$ and $\delta \to 0$.

For the signal whose representation in the frequency domain is sparse enough, in [16] Wang *et. al.* proved a lower bound for the signal-to-noise ratio loss after the decode session using CS:

$$SNR_{loss} \ \geq 10log_{10}\left(\frac{N}{K}\right). \quad (18)$$

Furthermore, the maximal sparsity is also estimated. In the perfect reconstruction ($SNR_{loss} < 0.5dB,\ N/K = 2$), Fourier representation is allowed to have at most 5 nonzero magnitudes. This implies the disadvantage of CS in compression.

ORTHOGONAL MATCHING PURSUIT (OMP)

Suppose that s is an arbitrary M-sparse signal in $\mathbb{R}^N$, and let $\{x_1, x_2, \dots, x_K\}$ be a family of K measurement vectors (for instance, the equation (2)). Form an $K \times N$ matrix

$$S = AF^{-1}, \quad (19)$$

whose rows are the measurement vectors, and observe that the K measurements of the signal can be collected in an K-dimensional data vector $v = S \bullet s$. Here $\hat{s}$ is replaced by s to put it more simply. We refer to S as the measurement "sensing" matrix and denote its columns by $\{\varphi_j, \varphi_2, \dots, \varphi_N\}$.

As we mentioned, it is intuitively to think of signal recovery as a problem dual to sparse approximation. Assume that s has only M nonzero components, the data vector $v = S \bullet s$ is a linear combination of columns from S where nonzero components of s have a bigger contribution to the measurement vector. Therefore, to identify the ideal signal s we need to determine which columns of S participate in the measurement vector v. The idea behind the algorithm is to greedily identify these columns. We choose the column of S at each iteration that is most strongly correlated with the remaining part of v. Finally, we subtract off its contribution to v and iterate on the residual. It is expected that after M iterations, the algorithm will have identified the correct set of columns.

Input: i) $K \times N$ sensing matrix S, ii) K-dimensional data vector v, iii) the sparsity M of the ideal signal and Φ_0 is initialized as an empty construction matrix.
Output: i) An estimate $\hat{s} \in \mathbb{R}^N$ for the ideal signal, and Λ_M containing M elements from $\{1, 2, \dots, N\}$.

<u>Algorithm:</u>

1. Initialize $r_0 \leftarrow v$, $\Lambda_0 \leftarrow \phi$, $t \leftarrow 1$.
2. Solve

$$\lambda_t \leftarrow argmax_{j=1,2,\dots,N} |\langle r_{t-1}, \varphi_j \rangle|,$$

where φ_j is j-th column of the sensing matrix.

3. $\Lambda_t \leftarrow \Lambda_{t-1} \cup \{\lambda_t\}$ and $\Phi_t \leftarrow [\Phi_{t-1} \varphi_{\lambda_t}]$.
4. Solve the least square problem

$$x_t \leftarrow argmin_x \|v - \Phi_t x\|_2.$$

5. Calculate the new approximation

$$a_t \leftarrow \Phi_t x_t,$$

$$r_t \leftarrow \mathrm{v} - \mathrm{a}_t.$$

6. $t \leftarrow t + 1$, return to Step 2 if $t < M$.
7. The approximation $\hat{s}$ for the ideal signal has nonzero indices at the components listed in Λ_M. The value of $\hat{s}$ in component λ_j is equal to the j-th component of x_t.

In other multiscale base (e.g. wavelets), the signals not only have few significant coefficients, but also those significant coefficients are well-organized in trees. Therefore, the Tree-based Orthogonal Matching Pursuit (TOMP)[3] is also often used to exploit this sparse tree representation as additional prior information for linear inverse problems with limited numbers of measurements. Although TOMP seems to be better than both OMP and BP in term of reconstruction quality when the number of measurements is limited and even better than OMP in computational efficiency, the TOMP algorithm is not discussed in this paper since it hasn't achieved the best performance.

3. BEST k-TERM APPROXIMATION

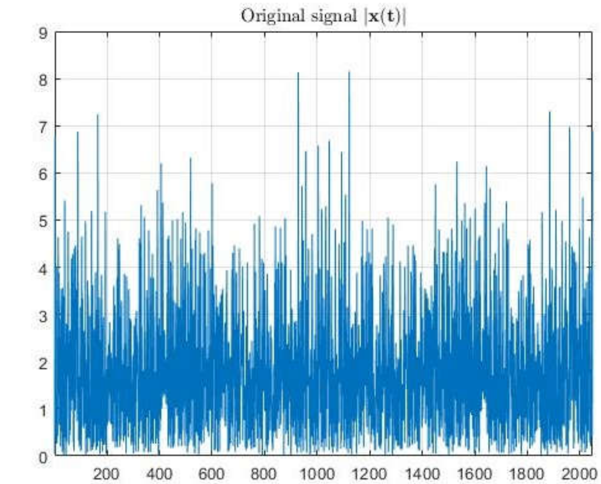
Compare to CS, BkTA is much simpler. The idea is after transforming the destination signal onto the Fourier domain, all the new magnitude values are zero out except k largest-magnitude components. The value k here should be chosen in order that the sub-sampling rate is less than twice since the compression algorithm is desired to save just at most half of the sampled data. The original signal is recovered simply by taking the inverse Fourrier transform of this pattern.

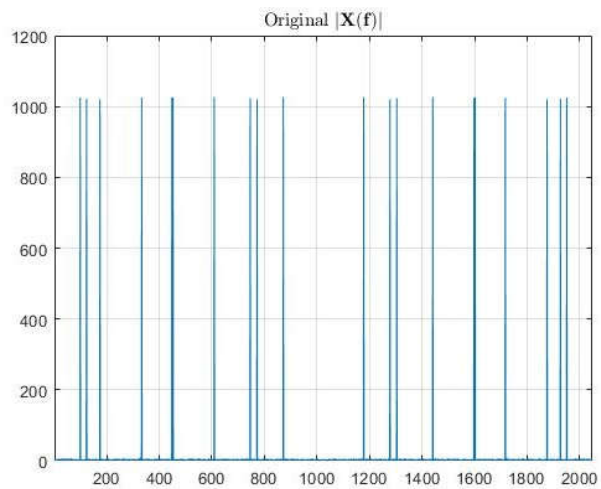
Algorithm:

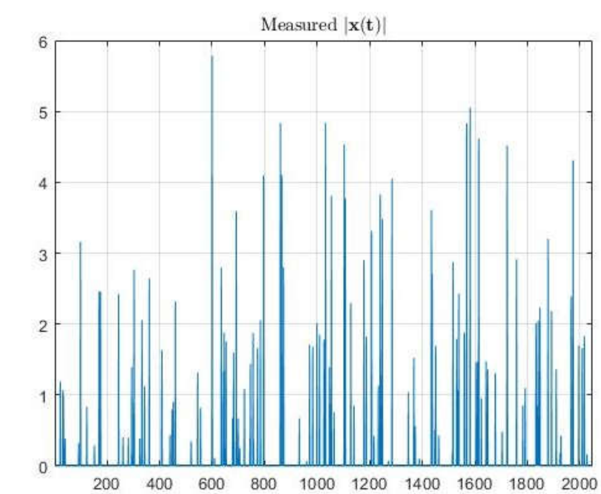
1. $x \leftarrow F \bullet s$.
2. $x \leftarrow x$ after keeping just k largest values.
3. $\hat{s} \leftarrow F^{-1} \bullet x$.

4. SIMULATION

In this section, we provide some numerical results for the signal compression using CS and BkTA. The scenario includes testing the CS and BkTA frameworks so that the sampling rate is reduced to a half. The signal-to-noise ratio is also taken into account since it dies out along the falling down of data recorded. First, the signals are initially generated normally to have a sparsity in Fourrier domain. The implementation follows the BP and/or OMP methods proposed before, respectively. The SNR loss here expectly depends on i) the sparsity, ii) the sampling rate and iii) the algorithm.







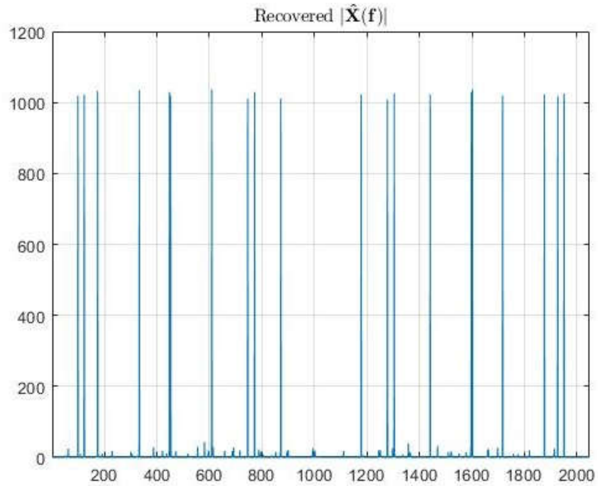


Figure 1: Sparse reconstruction using OMP

The signal: 20 frequency tones

$(N = 2048;\ K = 1024;\ sparsity\ 2\%;\ \ SNR_{loss} = 2.2804\ dB)$

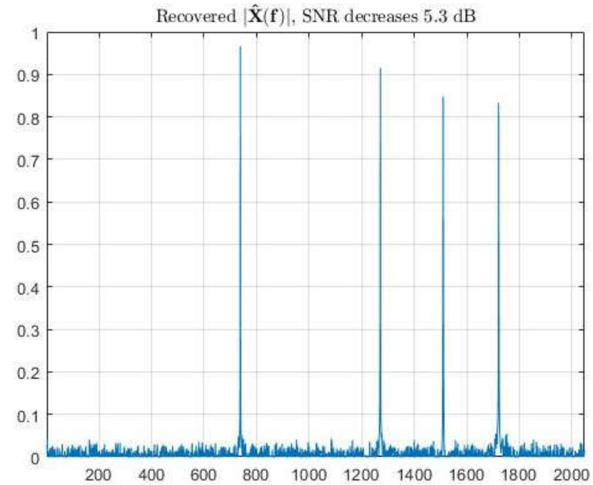
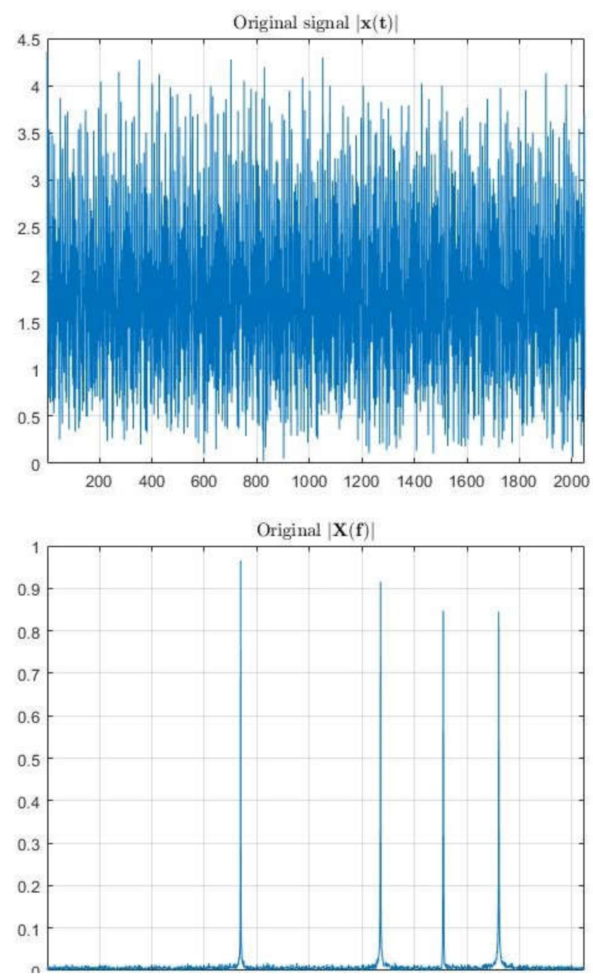
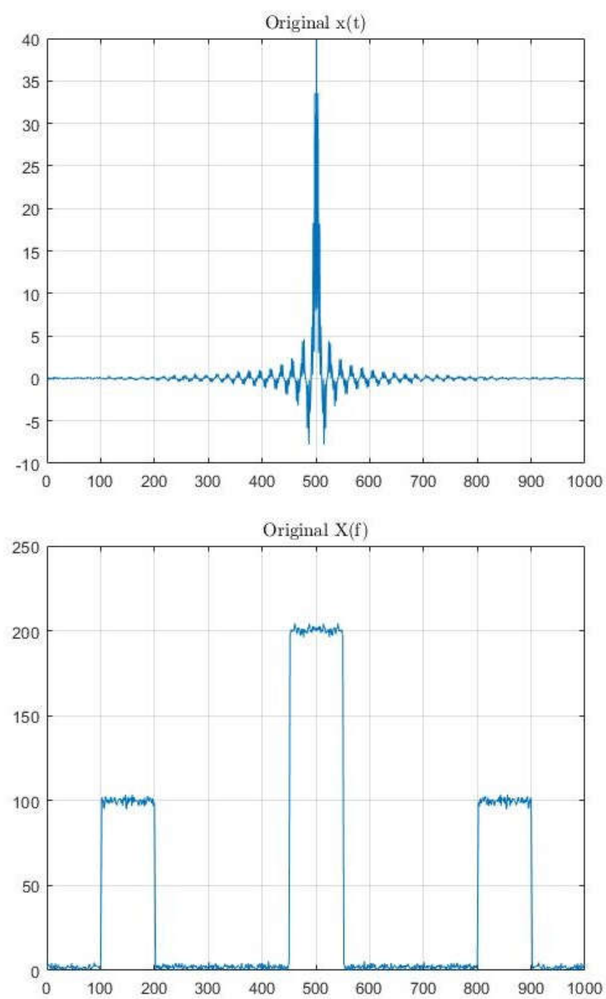


Figure 2: Sparse reconstruction using OMP

The signal: 50 MHz, down sampled.

$(N = 2048;\ K = 1024;\ sparsity\ 5\%;\ \ SNR_{loss} = 5.31\ dB)$





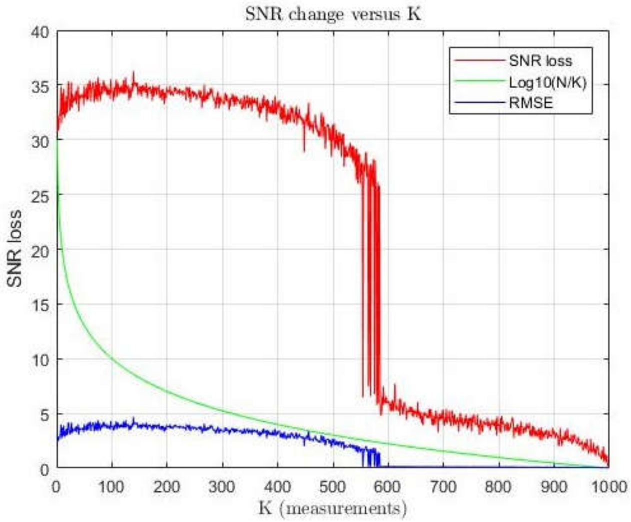


Figure 3: SNR loss versus measurement decreasing

The signal: 3 bands in frequency domain.

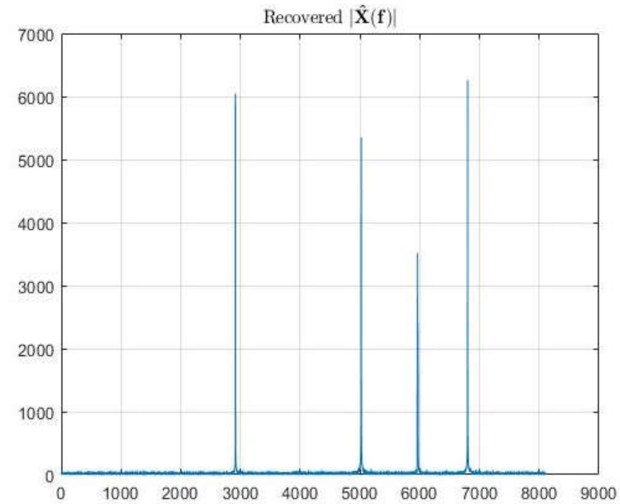
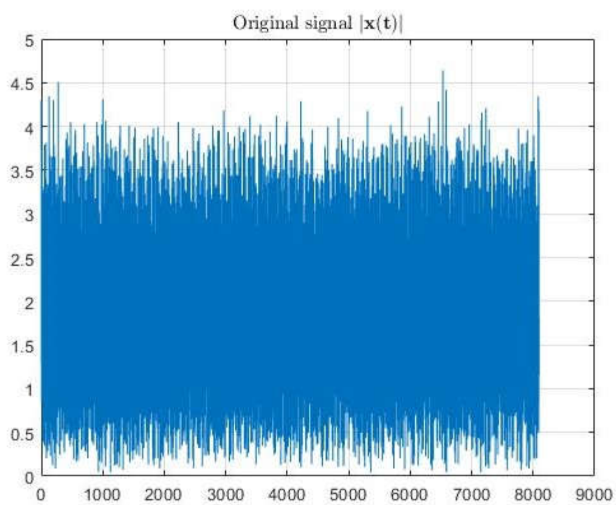
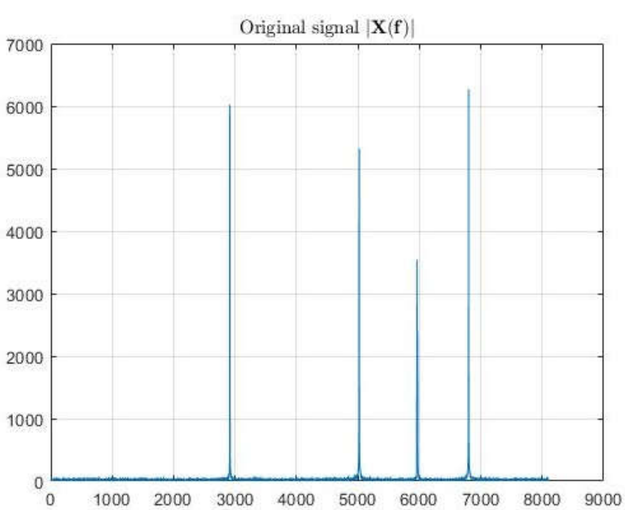


Figure 4: Sparse reconstruction using BkTA

The signal: 50 MHz, down sampled.

$(N = 8096;\ K = 4096;\ sparsity\ 5\%;\ \ SNR_{loss} = \ -0.6720\ dB)$





5. CONCLUSION

It could be seen from the simulation given in the CS part that the SNR loss is quite approximately equal to $10log_{10}\left(\frac{N}{K}\right)$ with respect to a suitable level of sparsity. As though CS has numerous applications in the image compression, i.e., 2-dimensional signals, for one-dimensional signals currently it is quite difficult to apply under the SNR loss benchmarks. Wavelet transforms allow us to see its representation but not all the characteristics except the edge magnitudes detection.

Experiments on the simulation show that BkTA is much efficient than CS. Nonetheless, it still requires a lot of works in improving the paradigms. For the future work, reducing the computational cost of the algorithm by optimizing the calculations in the BP, OMP and BkTA is preferable.

Table 1: Comparison between the two methods.

	Compressed sensing	BkTA
Encode	Fast	Fast
Decode	Slow	Fast
Capacity	Upper-medium	Lower-medium
SNR loss	Small	Very small
RMSE	Small	Small
Level	Medium	High

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