Determination of volumetric abundance of intimate mixture using Bayesian MCMC

Frédéric Schmidt, Bikram Koirala, Member and François Andrieu

Abstract—@ 2024 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

The quantitative estimation of volumetric abundance of powder mixture is the basis of quantitative remote sensing analysis. Here we propose to analyze a unique laboratory measurements set, with precise composition, grain size, and volumetric abundance. We first propose a method to estimate the optical constant of materials, knowing the pure endmember spectra and their grain size. Then, we propose a method to transfer the measurement uncertainties to the volumetric abundance, based on the Bayesian approach and the full Hapke radiative transfer model. Using this approach, we are able to estimate grain size,



volumetric abundance, and surface roughness. The results show that this approach is able to well estimate the correct volumetric abundance with an uncertainty of 23% and grain size with a ratio uncertainty of 3.0, i.e. uncertainties in log10(grain size)=0.48. The numerical cost of the MCMC is quite large (a few minutes per spectra) but still reasonable to treat a hyperspectral image with the gain of robust handling of non-linearities and propagating uncertainty.

Index Terms—radiative transfer, hyperspectral, intimate mixture

I. INTRODUCTION

D EPENDING on the scope, remote sensing spectroscopy and hyperspectral images can be interpreted in multiple ways, such as classification, segmentation, anomaly detection, and quantification on surface or atmosphere medium characteristics. This technique has been widely developed for the Earth but also for exploration of the Solar System. Here, we will focus on quantitatively estimating the microphysical parameters of materials such as their volumetric abundance, grain size, roughness, grain shape, and porosity from the measured spectral reflectances. The main difficulty arises because of the non-linearity of the radiative transfer which makes the problem difficult to solve with the usual gradient descent method. Even though some strategies have been proposed in planetary science to tackle this challenge, the scientific community lacks validation on real data.

Usually, tests of the radiative transfer are done in a reverse way. For instance, one common approach involves estimating the spectral reflectance of spherical granular materials using models and validating it by comparison with experimental data [1]. In this work, we will test the ability of a quantitative spectroscopic analysis scheme to estimate the correct parameters of interest (mainly abundances) by comparison with realistic laboratory experiments.

The applied mathematician community developed many

linear unmixing methods [2], including constraints, such as positivity, sparsity, endmember variability. These classes of methods are very interesting to provide a quick look at a complex hyperspectral dataset with a relatively fast computation time. Unfortunately, light-matter interaction is strongly nonlinear, preventing accurate abundance estimation. A second generation of approximated non-linear methods have been developed [3], [4], such as bilinear model, post-linear model, kernel approach with simplification of the radiative transfer, that would allow to approach the abundance estimation in specific condition (constant grain size, no roughness, no illumination condition effects) in relatively short computation complexity. However, these classes of methods suffer from a lack of generalization.

We will focus on the Hapke radiative transfer theory [5], which has the main advantage of being semi-analytic and thus very fast to compute. This model includes the photometrical effect of the BRDF (Bi-directional Reflectance Distribution Function) due to shadows produced by roughness. However, this model has limitations in the description of the light-matter interaction [6]. On the other hand, a more precise model requires much more computation time.

The Hapke model is highly non-linear and thus challenging to invert using traditional gradient descent methods [7]. To tackle non-linearities and estimate robust uncertainties, in this work, we will use a Monte Carlo Bayesian approach [8]. Such an approach has been initially developed for multi-angular photometric studies [9].

Some numerical validation tests of the Hapke theory have been performed [10], [11] but a more appropriate validation

F. Schmidt and F. Andrieu are within Université Paris-Saclay, CNRS, GEOPS, 91405, Orsay, France).

B. Koirala is with Imec-Visionlab, Department of Physics, University of Antwerp.

F. Schmidt is within Institut Universitaire de France

TABLE I DENSITY AND GRAIN SIZE (D50) OF THE MINERALS USED IN THIS STUDY.

Mineral	Density (g/cm ³)	Grain size (µm)
Al ₂ O ₃	3.98	3.5
CaO	3.34	2.7
Fe ₂ O ₃	5.26	0.8
SiO ₂	2.64	23
TiO ₂	3.89	0.5

test would be using real laboratory data. [12] proposes to validate different kinds of intimate mixtures by comparing observation and simulation spectra. The pioneering work of validation of abundance using real data has been performed by [13], [14] who found better than 5% error in weight %. Nevertheless, this work did not take into account the natural variability due to illumination, surface roughness and practical absolute calibration limitation [15]. Some approximations have been proposed to take into account the scaling effect of illumination but neglecting the full non-linearities [16]. Here instead, we choose to use the full Hapke model.

Only a few laboratory datasets are available to test the algorithms in real conditions with extensive volumetric abundance set [15], [17]. Unfortunately, the most complete one [17] lacks information about the grain size, so it cannot be applied in a quantitative validation case. We thus focused on a reduced but extremely well-characterized dataset [15] that encompasses the natural level variability.

In [18], [19], dedicated studies on quantitative validation of the roughness have been performed. On the other hand, in [20], the effect of roughness and grain shape on the spectral reflectances have been studied. Our work mostly focuses on the volumetric abundance, which is a crucial parameter for characterizing surface materials.

The aim of this article is to:

- present a method to quantify the microphysical parameters (volumetric abundance, grain size, and roughness).
- validate the approach on one of the most precise available laboratory data including natural variability.

II. LABORATORY DATA

The laboratory data set contains N = 63 spectra, including 14 pure mineral powders and 49 binary mixtures of mineral powders [15]. These mixtures were created using 5 selected pure mineral powders: Aluminum oxide (Al₂O₃), Calcium oxide (CaO), Iron oxide (Fe₂O₃), Silicon dioxide (SiO₂), and Titanium dioxide (TiO₂). All mineral powders have a white color in the visible (except for Iron oxide which is red). Furthermore, they vary in densities and grain sizes (see Table I).

Although there are 10 possible combinations of these powders, due to experimental constraints the dataset contains exclusively seven binary mixture combinations of minerals, namely: Al_2O_3 -SiO₂ (Al-Si), CaO-SiO₂ (Ca-Si), CaO-TiO₂ (Ca-Ti), Fe₂O₃-Al₂O₃ (Fe-Al), Fe₂O₃-CaO (Fe-Ca), Fe₂O₃-SiO₂ (Fe-Si), and SiO₂-TiO₂ (Si-Ti).

For each mineral combination, 7 distinct mixtures and the 2 pure endmembers were prepared. The total weight of each

mixture has been maintained at 10 g (with the scale accurate to 0.001 g). The mass weight was systematically incremented for the first mineral in the mixture by a fixed step of 1.25 g, ranging from 1.25 to 8.75 g. The weight fraction has been then converted to a volume fraction, based on the known density.

To ensure a homogeneous mixture, the mineral powders were placed inside a glass container and mixed thoroughly by rotating the container for approximately five minutes. To measure the reflectance spectra, each powder was subsequently transferred into a round black sample holder with an interior diameter of 20 mm, a height of 5.5 mm, and an edge thickness of approximately 3 mm. The sample holder was filled with the mixture, compacted, and smoothed using a stamp compactor.

The samples were scanned using an AgriSpec spectrometer manufactured by Analytical Spectral Devices (ASD) that comprises 1500 spectral bands, ranging from 1000 to 2500 nm with a step size of 1 nm. The wavelength range was adjusted to match that of the hyperspectral camera, resulting in 537 bands. The full dataset was calibrated using a spectralon. All details about this experimental dataset can be found in [15].

The spectral reflectance is vector noted R with a dimension $N_{\lambda} = 537$. To estimate the uncertainties of measurements from this data, the radiance of the Spectralon (white calibration panel) measured throughout the experiments was considered. In this work, the relative reflectance of the sample is determined by the ratio of the radiance of the sample to the radiance of the white calibration panel. We found that the standard deviation σ_R for this dataset is 0.7 %.

The provided dataset is particularly difficult to handle, without renormalization [15] since several measurements of the same pure endmember are reproduced with a significant absolute level variability (see Figure 1 (h)). We attribute this effect to the powder preparation and compaction which may vary significantly despite the meticulous care of the operator. In addition, the observation and illumination geometries may have changed to produce this spurious effect. Last, even using a Spectralon, the dataset may still include uncertainties due to the absolute calibration process. In Figure 1, the spectral reflectances of binary mixtures composed of Aluminum oxide, Calcium oxide, Iron oxide, Silicon dioxide, and Titanium dioxide are shown. As can be observed, the spectral features of these minerals are clearly visible and gradually change when the fractional abundance (mass ratios) of each mineral changes in the mixture.

III. METHOD

A. Hapke radiative transfer model

1) Theory: We used the semi-analytical reflectance model from [5], which is a good compromise between physical realism and efficient computation time. Based on this model, the spectral reflectance of the material R_s can be modeled as:

$$R_{s}(\mu_{0},\mu,g) = \frac{\omega}{4\pi} \frac{\mu_{0}}{(\mu_{0}+\mu)} \Big((1+B(g)) P(g) + H(\mu_{0})H(\mu) - 1 \Big) S(\mu_{0},\mu,g)$$
(1)

Where μ_0 is the cosine of the incidence angle, μ is the cosine of the emission angle, g is the phase angle, ω is the



Fig. 1. Spectra of binary mixtures. a) Al-Si; b) Ca-Si; c) Ca-Ti; d) Fe-Al; e) Fe-Ca; f) Fe-Si; g) Si-Ti; and h) Endmembers.

single-scattering albedo, P(g) is the particle phase function, B(g) is the opposition effect, H represents the multiple scattering effects and S the shadowing effect depending on the macroscopic roughness θ_0 . To simplify the expression, we assume that multiple scattering is isotropic.

The single scattering albedo of the material can be modeled as the ratio of the mean scattering cross section $\langle \sigma_s \rangle$ and the mean extinction cross-section $\langle \sigma_e \rangle$:

$$\omega = \frac{\langle \sigma_s \rangle}{\langle \sigma_e \rangle} \tag{2}$$

with $<\sigma_e>$ the mean geometrical cross-section of the grains and

$$<\sigma_s>=\sum_k X'_k\sigma_{sk}$$
 (3)

Each type of grain k is present in the surface with the numerical proportions X'_k . Volumetric fractional abundances X_k are computed by rescaling to the total volume of grain:

$$X_k = \frac{X'_k \phi_k^3}{\sum_k X'_k \phi_k^3} \tag{4}$$

A type of grain is defined by its average particle diameter ϕ_k and its chemical composition. For a type k grain, the scattering cross section σ_{sk} is defined as:

$$\sigma_{sk} = \sigma_k \left(S_e + (1 - S_e) \frac{(1 - S_i)}{(1 - \Theta_k S_i)} \right)$$
(5)

where $\sigma_k = \pi \times \phi_k^2/4$ is the particle geometric cross-section, and S_e and S_i are respectively the external and internal particle reflection coefficients, which are obtained by integrating the Fresnel reflection coefficients over the surface of the grain from the optical constant n and κ (see section III-A.2). Θ_k is the transmission coefficient of a type k grain, depending on optical constant and grain size. All details can be found in [5].

As a summary, the Hapke model can be written as:

$$R_s = \mathcal{H}(\theta_0, \phi, X) \tag{6}$$

where ϕ and X represent the grain-sizes and numerical proportions of the different types of grains constituting the surface.

2) Optical constant: Shkuratov proposed a theory to estimate the imaginary part κ of the optical constant (absorption coefficient) from the reflectance spectra R. This theory assumes that the real part n (refractive indices) and the grain size ϕ of the material are known as priori. We assume that the refractive index for minerals is n = 1.9, but we conduct the same experiment for n = 1.5 in order to demonstrate the minor effect of this assumption. The grain size ϕ is experimentally known (see Table I).

$$\kappa = \mathcal{S}(R, n, \phi_s) \tag{7}$$

This approach has been first proposed to estimate the Martian dust properties [21], [22]. The complete benchmark of the effects of the grain size (mean free path length), filling factor, and real optical index on the final result is relatively limited, as discussed in [21].

B. Bayesian MCMC inversion

We used a Bayesian approach to propagate the uncertainties from the measurement (reflectance R) to the parameters. This operation is sometimes called inversion or assimilation. In [7], we demonstrated that the usual gradient-descent method is not helpful because the non-linearities are so strong that the results mainly depend on the initialization.

The MCMC approach consists of sampling the posterior distribution using the Bayesian rules. The method not only explores the parameter space and looks for all acceptable solutions, but also determines the best set of parameters to fit the data. From this collection of acceptable spectra/parameters, we can look at the marginalized posterior distribution of each parameter for a typical fit.

1) prior: The prior distribution of the parameters are :

- The volumetric fractional abundances X_k should be positive and sum to one $(\sum_{k=1}^2 X_k = 1)$. We use the parameter $X = X_1 = 1 X_2$ to describe a binary mixture with its probability p(X) = 1 for $X \in [0, 1]$.
- The grain-size diameter $\phi_k = 10^{D_k}$ of each endmember *i* is assumed to follow a log-uniform distribution, so that $D_k = \log 10(\phi_k)$ follows a uniform distribution $p(D_k) = 1/(D_{max} - D_{min})$ for $D_k \in [D_{min}, D_{max}]$. We set $D_{min} = -1$ and $D_{max} = 3$.
- The macroscopic surface roughness θ_0 for which we assume a uniform distribution $p(\theta_0) = \frac{1}{45}$ for $\theta_0 \in [0, 45]$

2) MCMC sampling: We used the snooker algorithm [23] implemented in the mc3 package [24] to sample the posterior distribution, knowing the observation R and its uncertainties σ_R .

Since the solution is a probability density function (PDF) that may differ from a Gaussian, neither the best fit, nor the maximum posterior, nor the the average PDF may be representative of the full solution. We choose to summarize the posterior PDF as the median (noted \bar{X} , \bar{D} and $\bar{\theta_0}$) and the 68.3% quantiles low (noted X, D and $\tilde{\theta_0}$) and high bounds (noted \hat{X} , \hat{D} and $\hat{\theta_0}$).

IV. RESULTS

The quantitative results are presented through the root mean squared error (RMS) calculated between the estimated (\bar{X}) and the ground-truth abundance (X).

$$\mathbf{RMS} = \sqrt{\frac{\sum (\bar{X} - X)^2}{N} \times 100} \tag{8}$$

The average over N can be either over the full dataset (N = 63) or restricted to the binary mixture including the pure endmember (N = 9). The definition is trivially extended to the grain size $D = \log 10(\phi)$, except that the number of values is doubled since two minerals are present for each experiment.

A. Example

Figure 2 presents a typical result. This particular one is for a CaSi mixture with X=57.0% Ca. The best fit is extremely close to the real observation. In addition, due to the uncertainties on the reflectance spectra, a range of solutions is acceptable.



Fig. 2. Example of a fit for CaSi mixture



Fig. 3. Posterior PDF of the same case as Fig. 2 for the four unknown parameters: roughness θ_0 , volumetric proportion, X_{Ca} , grains size $\log 10(\phi_{Ca})$ and $\log 10(\phi_{Si})$. The median values are indicated by a dashed line.

Figure 3 shows the corresponding posterior PDF for the four parameters that are unknowns: roughness θ_0 , volumetric proportion, X_{Ca} , grains size $\log 10(\phi_{Ca})$ and $\log 10(\phi_{Si})$. All marginal PDFs are well constrained with a bell shape, except the $\log 10(\phi_{Si})$ which is less constrained. The bivariate PDF indicates the relationship between the parameters. For instance, roughness seems highly correlated with the X_{Ca} .

B. Results

Figure 4 shows the estimation of the abundance for all the cases. It shows that the trend is well reproduced in all cases. FeSi and SiTi have significantly higher errors in abundance retrieval because neither Si nor Ti have strong spectral features. The best results are obtained when both endmembers have a strong specific spectral feature (such as Fe and Ca in FeCa).

Table II presents the RMS for volumetric abundance and grain size. The departure from the volumetric abundance true value depends on the binary mixture but the errors remain low in general with 23% on average. Given the complexity of the dataset, we consider that this value is acceptable. It is better than the linear and the approximated Hapke manifold [15]. The geodesic supervised unmixing(GSU) [15] can reach less than 1% uncertainties in abundances, but at the cost of fitting this particular dataset. Here the model is set for whatever

TABLE II

RMS ERROR FOR VOLUMETRIC ABUNDANCE (IN %) AND GRAIN SIZE (IN % $D = \log 10(\phi_k)$). The Global RMS is 23.0% (23.3%) for X and 48.9% (44.6%) for D. When the grain size is known, the RMS for X reduces to 21.1% (21.4%). All the results are with the assumption of n = 1.9, except in parenthesis FOR n = 1.5

Mixture	RMS X	RMS D	RMS X knowing D
AlSi	17.4 (18.7)	39.4 (33.7)	6.4 (6.7)
CaSi	12.6 (11.6)	39.9 (56.6)	23.9 (26.9)
CaTi	29.4 (29.9)	136.9 (113)	29.8 (31.4)
FeAl	35.8 (36.6)	73.6 (84.6)	31.0 (30.2)
FeCa	9.1 (10.2)	66.2 (80.4)	15.9 (15.5)
FeSi	32.8 (33.9)	65.8 (74.4)	25.8 (25.8)
SiTi	23.4 (20.1)	103.6 (76.2)	28.9 (23.0)

compositional mixture, abundance, grain size, roughness, and illumination/acquisition geometry conditions. The only fitting parameters are the optical constant κ and n spectra for each pure endmember.

The largest error occurs in FeAl and FeSi because the absolute level uncertainties of the reflectance are the worst constrained in these cases. As seen in Fig. 1 (h) Fe, Al, and Si have the highest pure endmembers variability.

For the grain size, the RMS in *D* is 48% on average. For a single parameter, the error is simply the absolute difference between the true and estimated values. Consequently, the error in *D* results in a logarithmic ratio in the true and estimated ϕ . The error in ϕ can then be computed by taking $10^{0.48} = 3.0$. This implies that the estimated grain size $\bar{\phi}$ falls within the range $\phi/3.0 < \bar{\phi} < \phi \times 3.0$, which is satisfactory considering the extensive range of grain sizes.

The last column of Table II presents the RMS for volumetric abundance assuming that the grain size is known (and thus not estimated). The global RMS on abundance is only slightly improving and on some binary mixtures, the values are getting worse. This behavior is not expected because usually the more information you provide, the better the results. This is not the case here because of the lack of absolute-level calibration. In other words, the effect of grain size can accommodate the uncertainties in absolute levels.

Finally, Table II also presents the results for two assumptions of the real index of the optical constant n = 1.9 by default, but also n = 1.5 in parenthesis. The results are very similar, demonstrating that this parameter is minor for our retrieval.

V. DISCUSSION

The methodology does not require the optical constant but only pure endmember spectra. In real remote sensing data, this is often not present in the scene. It may be possible to estimate endmembers by utilizing blind spectral unmixing methods such as the one presented in [25]. On the other hand, the grain size of the material can be estimated by utilizing thermal inertia [26]. Of course, if the optical constants are available, this first step can be bypassed.

Remote sensing data on the field may not observe powders. To evaluate the performance of the proposed method on realistic samples, an extensive hyperspectral dataset with accurate



Fig. 4. True versus estimated volumetric abundance for all binary mixtures, estimating the four following parameters: roughness θ_0 , volumetric proportion, *X*, grains size log $10(\phi_1)$ and log $10(\phi_2)$. The median and the 68.3% quantiles are plotted as the error bar.

ground truth information is required. The literature generally lacks systematic ground truth experimental measurement to validate the developed numerical approaches. In future works, hyperspectral datasets obtained from extensive geological samples will be analyzed. In addition, the effect of grain size will be considered with more attention.

VI. CONCLUSION

- We proposed a methodology to characterize powder samples from the shortwave infrared reflectance dataset. In the first step, the optical constant of the pure mineral was estimated by utilizing its spectral reflectance. We then estimated the volumetric abundance, roughness, and grain size using a Bayesian Monte Carlo.
- This approach is able to constrain abundance with an RMS of 23% and a ratio on grain size error of around 3.0. Both errors in abundance and grain size are no-

tably favorable compared to previous literature, especially considering the challenges posed by non-linear radiative transfer.

- The methodology is able to propagate the uncertainties from the measured reflectance to the parameters (volumetric abundance, grain size and roughness) in a consistent manner.
- One of the limitations of the approach is the large computation time. Nevertheless, since the Hapke radiative transfer model is closed-form semi-analytical, the computation time is reasonable to treat a full hyperspectral cube (a few minutes per spectra).

ACKNOWLEDGMENT

We acknowledge support from the "Institut National des Sciences de l'Univers" (INSU), the "Centre National de la Recherche Scientifique" (CNRS) and "Centre National d'Etudes Spatiales" (CNES) through the "Programme National de Planétologie". Bikram Koirala is a postdoctoral fellow of the Research Foundation Flanders, Belgium (FWO: 1250824N-7028)

REFERENCES

- [1] B. W. Hapke, M. K. Shepard, R. M. Nelson, W. D. Smythe, and J. L. Piatek, "A quantitative test of the ability of models based on the equation of radiative transfer to predict the bidirectional reflectance of a well-characterized medium," *Icarus*, vol. 199, no. 1, pp. 210–218, Jan. 2009. [Online]. Available: http://www.sciencedirect.com/science/ article/B6WGF-4TMSNWP-4/2/e453e1a339899062eedb00bcd2fa27d0
- [2] J. Bioucas-Dias, A. Plaza, N. Dobigeon, M. Parente, Q. Du, P. Gader, and J. Chanussot, "Hyperspectral unmixing overview: Geometrical, statistical, and sparse regression-based approaches," *Selected Topics in Applied Earth Observations and Remote Sensing, IEEE Journal of*, vol. 5, no. 2, pp. 354–379, 2012.
- [3] R. Heylen, M. Parente, and P. Gader, "A review of nonlinear hyperspectral unmixing methods," *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing*, vol. 7, no. 6, pp. 1844–1868, jun 2014.
- [4] N. Dobigeon, J.-Y. Tourneret, C. Richard, J. Bermudez, S. McLaughlin, and A. Hero, "Nonlinear unmixing of hyperspectral images: Models and algorithms," *Signal Processing Magazine, IEEE*, vol. 31, no. 1, pp. 82– 94, Jan 2014.
- [5] B. Hapke, *Theory of Reflectance and Emittance Spectroscopy*, 2nd ed. Cambridge University Press, 2012.
- [6] Y. Shkuratov, V. Kaydash, V. Korokhin, Y. Velikodsky, D. Petrov, E. Zubko, D. Stankevich, and G. Videen, "A critical assessment of the hapke photometric model," *Journal of Quantitative Spectroscopy* and Radiative Transfer, vol. 113, no. 18, pp. 2431–2456, Dec. 2012. [Online]. Available: http://www.sciencedirect.com/science/article/ pii/S0022407312001926
- [7] F. Andrieu, F. Schmidt, G. Cruz-Mermy, I. Belgacem, and T. Cornet, "Benchmarking Bayesian methods for spectroscopy," in *European Planetary Science Congress*, Sep. 2022, pp. EPSC2022–502.
- [8] F. Schmidt and J. Fernando, "Realistic uncertainties on hapke model parameters from photometric measurement," *Icarus*, vol. 260, pp. 73

 93, 2015. [Online]. Available: http://www.sciencedirect.com/science/ article/pii/S0019103515002936
- [9] J. Fernando, F. Schmidt, X. Ceamanos, P. Pinet, S. Douté, and Y. Daydou, "Surface reflectance of Mars observed by CRISM/MRO: 2. Estimation of surface photometric properties in Gusev Crater and Meridiani Planum," *Journal of Geophysical Research (Planets)*, vol. 118, pp. 534–559, Mar. 2013. [Online]. Available: http: //dx.doi.org/10.1029/2012JE004194
- [10] H. Zhang and K. J. Voss, "On hapke photometric model predictions on reflectance of closely packed particulate surfaces," *Icarus*, vol. 215, no. 1, pp. 27–33, Sep. 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0019103511002946

- [11] M. Ciarniello, F. Capaccioni, and G. Filacchione, "A test of hapke's model by means of monte carlo ray-tracing," *Icarus*, vol. 237, no. 0, pp. 293–305, Jul. 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0019103514002371
- [12] M. Ciarniello, L. V. Moroz, O. Poch, V. Vinogradoff, P. Beck, B. Rousseau, I. Istiqomah, R. Sultana, A. Raponi, G. Filacchione, D. Kappel, A. Pommerol, S. E. Schröder, C. Pilorget, E. Quirico, V. Mennella, and B. Schmitt, "Vis-ir spectroscopy of mixtures of water ice, organic matter, and opaque mineral in support of small body remote sensing observations," *Minerals*, vol. 11, no. 11, p. 1222, 2021-11.
- [13] J. F. Mustard and C. M. Pieters, "Quantitative abundance estimates from bidirectional reflectance measurements," vol. 92, no. B4.
- [14] T. Hiroi and C. M. Pieters, "Estimation of grain sizes and mixing ratios of fine powder mixtures of common geologic minerals," *J. Geophys. Res.*, vol. 99, pp. –, 1994. [Online]. Available: http://dx.doi.org/10.1029/94JE00841
- [15] B. Koirala, Z. Zahiri, A. Lamberti, and P. Scheunders, "Robust supervised method for nonlinear spectral unmixing accounting for endmember variability," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 59, no. 9, pp. 7434–7448, 2021.
- [16] L. Drumetz, J. Chanussot, and C. Jutten, "Spectral unmixing: A derivation of the extended linear mixing model from the hapke model," *IEEE Geoscience and Remote Sensing Letters*, vol. 17, no. 11, pp. 1866–1870, nov 2020.
- [17] B. Koirala, B. Rasti, Z. Bnoulkacem, A. d. L. Ribeiro, Y. Madriz, E. Herrmann, A. Gestels, T. D. Kerf, S. Lorenz, M. Fuchs, K. Janssens, G. Steenackers, R. Gloaguen, and P. Scheunders, "A multisensor hyperspectral benchmark dataset for unmixing of intimate mixtures," *IEEE Sensors Journal*, pp. 1–1, 2023.
- [18] S. Labarre, C. Ferrari, and S. Jacquemoud, "Surface roughness retrieval by inversion of the hapke model: A multiscale approach," *Icarus*, vol. 290, pp. 63–80, Jul 2017. [Online]. Available: http: //dx.doi.org/10.1016/j.icarus.2017.02.030
- [19] S. Labarre, S. Jacquemoud, C. Ferrari, A. Delorme, A. Derrien, R. Grandin, M. Jalludin, F. Lemaître, M. Métois, M. Pierrot-Deseilligny, E. Rupnik, and B. Tanguy, "Retrieving soil surface roughness with the hapke photometric model: Confrontation with the ground truth," *Remote Sensing of Environment*, vol. 225, pp. 1–15, 2019.
- [20] A. Souchon, P. Pinet, S. Chevrel, Y. Daydou, D. Baratoux, K. Kurita, M. Shepard, and P. Helfenstein, "An experimental study of hapke's modeling of natural granular surface samples," *Icarus*, vol. 215, no. 1, pp. 313–331, Sep. 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0019103511002405
- [21] S. Douté, B. Schmitt, Y. Langevin, J.-P. Bibring, F. Altieri, G. Bellucci, B. Gondet, F. Poulet, and the MEX OMEGA team, "South Pole of Mars: Nature and composition of the icy terrains from Mars Express OMEGA observations," *Planetary and Space Science*, vol. 55, pp. 113–133, Jan. 2007.
- [22] F. Andrieu, F. Schmidt, S. Douté, and E. Chassefière, "Ice state evolution during spring in richardson crater, mars," *Icarus*, vol. 315, pp. 158–173, nov 2018.
- [23] C. J. F. ter Braak and J. A. Vrugt, "Differential evolution markov chain with snooker updater and fewer chains," *Statistics and Computing*, vol. 18, no. 4, pp. 435–446, 2008.
- [24] P. Cubillos, J. Harrington, T. J. Loredo, N. B. Lust, J. Blecic, and M. Stemm, "ON CORRELATED-NOISE ANALYSES APPLIED TO EXOPLANET LIGHT CURVES," *The Astronomical Journal*, vol. 153, no. 1, p. 3, dec 2016.
- [25] B. Rasti, B. Koirala, and P. Scheunders, "Hapkeenn: Blind nonlinear unmixing for intimate mixtures using hapke model and convolutional neural network," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 60, pp. 1–15, 2022.
- [26] B. Gundlach and J. Blum, "A new method to determine the grain size of planetary regolith," *Icarus*, vol. 223, no. 1, pp. 479–492, Mar. 2013. [Online]. Available: http://www.sciencedirect.com/science/article/ pii/S0019103512004939