Parameter estimation from magnitude MR images

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Abstract

This paper deals with the estimation of model-based parameters, such as the noise variance and signal components, from magnitude Magnetic Resonance (MR) images. Special attention has been paid to the estimation of T_1- and T_2-relaxation parameters. It is shown that most of the conventional estimation methods, when applied to magnitude MR images, yield biased results. Also, it is shown how the knowledge of the proper probability density function of magnitude MR data (i.e., the Rice distribution) can be exploited so as to avoid (or at least reduce) such systematic errors. The proposed method is based on Maximum Likelihood (ML) estimation.

1 Introduction

Complex valued raw Magnetic Resonance (MR) images are known to be corrupted with zero mean Gauss distributed noise with equal variance [1, 2]. After an inverse Fourier transformation, the real and imaginary data are still Gauss distributed due to the orthogonality and the linearity of the Fourier transform [3]. Commonly, however, the MR data are non-linearly transformed into magnitude (and/or phase) data, as these data are more directly related to physical quantities such as the pseudo-proton density, perfusion, diffusion, or flow. Conventional magnitude data, which are computed from two independent Gauss distributed variables (the real and imaginary variables), can be shown to be Rice distributed [4]. When the magnitude data are computed from an arbitrary number of Gauss distributed variables (e.g., several real and imaginary variables), these data can be shown to be governed by a generalized Rice distribution [5]. This is the case for phase contrast magnitude MR data, which are encountered in angiographic MR imaging.

First, the estimation of the noise variance is considered. In the image processing literature, most of the applied methods to estimate the image noise variance assume Gauss distributed noise [6, 7, 8, 9]. However, as stated above, magnitude MR data are no longer Gauss but Rice distributed. In this work, it is demonstrated how the properties of the (generalized) Rice distribution can be exploited to estimate the image noise variance from magnitude MR data. Although several methods were proposed for noise estimation from two realizations of the same image ([10, 11, 12]), in this work, we will concentrate on the problem of the estimation of noise from a single image as multiple images are often not available in practice.
In MR imaging, various methods were reported on the estimation of the noise variance or the noise standard deviation from non-signal background regions [13, 14, 15], where the data are

known to be Rayleigh distributed. In this work, the properties of these estimators are discussed in detail. It is shown that methods that are based on Maximum Likelihood estimation perform best. Also, an extension to generalized Rayleigh distributed data is proposed.

Furthermore, model-based estimation of signal parameters from (generalized) Rice distributed data is described, and an application to T_1 and T_2 parameter estimation is discussed in detail. Estimation of relaxation parameters has been a subject of considerable interest from the early years of Magnetic Resonance (MR) imaging. Both the spin-lattice relaxation parameter and the spin-spin-relaxation parameter T_2 give useful information about the interaction with the local environment, and play a major role in the establishment of image contrast.
Conventional relaxation parameter estimation techniques, applied to magnitude MR images, consist of (weighted) least squares fitting procedures, which are only optimal for Gauss distributed data [16]. However, magnitude MR data are Rice distributed. Recently, a paper was published on the use of the Rice distribution in the problem of estimating T_2 maps from magnitude MR data [17]. In that paper, the problem on the data distribution was recognized, but parameter estimation was still performed assuming Gauss distributed noise. The authors justified the use of least squares estimation by stating that at high signal-to-noise ratio (SNR), the Rice probability density function (PDF) approaches a Gauss PDF. Although this is true, a bias is introduced in the estimation procedure, which becomes more pronounced with decreasing SNR [18].
In the present work, a Maximum Likelihood (ML) estimation technique is proposed for optimal estimation of both the spin-lattice and the spin-spin relaxation time from a set of magnitude MR images. This choice is motivated by the fact that an ML estimator is known to be consistent and asymptotically most precise [19]. In the construction of the ML estimator, full use is made of the Rice distribution. The validity of the proposed method is checked by simulation experiments.

2 Noise estimation

In the following, various methods for the estimation of the noise variance and the noise standard deviation from magnitude MR data are discussed. Thereby, it will be assumed that the available data are obtained from computing the overall magnitude of K Gauss distributed, independent and hence uncorrelated [1] variables $\{x_k\}$:

$$M = \sqrt{\sum_{k=1}^{K} x_k^2} \tag{1}$$

(Remark that $K = 2$ for conventional Rice distributed magnitude MR data.) It can be shown that the magnitude variable M is governed by a generalized Rice distribution [5].

2.1 Estimation of the noise variance

The value of σ^2 can be estimated in many ways. Usually, σ^2 is estimated from the data points in a region of interest (ROI) in the background area, where the deterministic signal components of the variables $\{x_k\}$ are assumed to be zero. In such regions, the magnitude data are governed by a generalized Rayleigh distribution [5]:

$$p_M(M|\sigma^2) = \frac{2M^{K-1}}{(2\sigma^2)^{K/2}\Gamma\left(\frac{K}{2}\right)} \exp\left(-\frac{M^2}{2\sigma^2}\right) \epsilon(M) \quad . \tag{2}$$

Now, from the second moment of this distribution,

$$E\left[M^2\right] = K\sigma^2 \quad , \tag{3}$$

[1] Note that uncorrelatedness is only a fair assumption if no zero filling was performed prior to the Fourier transformation of the raw MR data.

an estimator of the noise variance can easily be derived from a spatial average of N magnitude variables $\{M_i\}$ in the ROI:

$$\widehat{\sigma^2} = \frac{1}{KN}\sum_{i=1}^{N} M_i^2 \quad , \tag{4}$$

It can easily be shown that (4) is an unbiased estimator of σ^2 with a variance equal to σ^4/N. In addition, the estimator is identical to the Maximum Likelihood estimator. This can be seen as follows.
First, we write the joint probability density function f_w of the set of N magnitude variables $\{M_i\}$ as:

$$f_w = \prod_{i=1}^{N} p_M(M_i|\sigma^2) \tag{5}$$

When numbers are substituted for the set of magnitude variables $\{M_i\}$, and σ^2 is regarded as a parameter, the joint probability density function is called the Likelihood function, given by:

$$L = \prod_{i=1}^{N} p_M(M_i|\sigma^2) \tag{6}$$

$$= \left(\sigma^2\right)^{-NK/2} \left(\prod_{i=1}^{N} \frac{2M_i^{K-1}}{2^{K/2}\Gamma\left(\frac{K}{2}\right)}\right) \prod_{i=1}^{N} \exp\left(-\frac{M_i^2}{2\sigma^2}\right) \quad , \tag{7}$$

where factors depending on σ^2 were grouped. Taking the logarithm, and leaving only the terms that depend on σ^2, we have:

$$\log L \sim -\frac{NK}{2}\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{N} M_i^2 \quad . \tag{8}$$

Maximization of (8) requires the first order derivative of L with respect to σ^2 to be zero:

$$\frac{\partial \log L}{\partial \sigma^2} = 0 \quad , \tag{9}$$

yielding the unbiased estimator given in Eq. (4). The estimator (4) indeed maximizes L, as the second derivative of L with respect to σ^2 is always negative. For all N, its variance equals the Cramér-Rao lower bound (CRLB), which is a lower bound on the variance of any unbiased estimator of σ^2. The CRLB of any function $g(\sigma^2)$ of σ^2 can be explicitly computed [19]. In case $g(\sigma^2)$ equals σ^2, we have:

$$\mathrm{CRLB}\left(g\left(\widehat{\sigma^2}\right)\right) = -\frac{\partial g(\sigma^2)}{\partial \sigma^2}\left(E\left[\frac{\partial^2 \log f_w}{\partial (\sigma^2)^2}\right]\right)^{-1}\frac{\partial g(\sigma^2)}{\partial \sigma^2}] \tag{10}$$

$$= -\left(E\left[\frac{NK}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{N} M_i^2\right]\right)^{-1} \tag{11}$$

$$= \frac{2\sigma^4}{NK} \quad . \tag{12}$$

2.2 Estimation of the noise standard deviation

One might be interested in the value of the standard deviation σ as well. Taking the square root of the ML estimate of σ^2 yields an estimator of σ:

$$\hat{\sigma}_{ML} = \sqrt{\frac{1}{KN}\sum_{i=1}^{N} M_i^2} \quad . \tag{13}$$

This estimator is identical to the ML estimator of σ, as the square root operation has a single valued inverse (cfr. Invariance property of ML estimators [20]). Its variance is approximately equal to:

$$\mathrm{Var}\left(\hat{\sigma}_{ML}\right) \simeq \frac{\sigma^2}{2NK} \quad . \tag{14}$$

The right hand side of Eq. (14) equals the CRLB. This can be seen from Eq. (10) with $g(\sigma^2) = \sqrt{\sigma^2}$:

$$\begin{aligned} \mathrm{CRLB}\left(\hat{\sigma}_{ML}\right) &= \frac{\partial\sqrt{\sigma^2}}{\partial\sigma^2}\mathrm{CRLB}\left(\widehat{\sigma^2}\right)\frac{\partial\sqrt{\sigma^2}}{\partial\sigma^2} & (15)\\ &= \frac{\sigma^2}{2NK} \quad . & (16)\end{aligned}$$

However, the estimator (13) is biased because of the square root operation. Its expectation value is approximately equal to:

$$E\left[\hat{\sigma}_{ML}\right] \simeq \sigma\left(1 - \frac{1}{4NK}\right) \quad . \tag{17}$$

Notice that this means that it is possible to apply a bias correction. This, however, would increase the variance of the estimator.
Another commonly used estimator of σ can be found by exploiting the knowledge that the Rice PDF turns into a Rayleigh PDF in image regions with no signal. Since the mean value of the generalized Rayleigh PDF is given by

$$E[M] = \sqrt{2}\sigma\frac{\Gamma\left((K+1)/2\right)}{\Gamma\left(K/2\right)} \quad , \tag{18}$$

an unbiased estimator of σ is given by:

$$\hat{\sigma}_c = \frac{\Gamma\left(K/2\right)}{\Gamma\left((K+1)/2\right)}\frac{1}{\sqrt{2}N}\sum_{i=1}^{N} M_i \quad . \tag{19}$$

The variance of this estimator is given by

$$\mathrm{Var}\left(\hat{\sigma}_c\right) = \frac{\sigma^2}{N}\left(\frac{K}{2}\left(\frac{\Gamma\left(K/2\right)}{\Gamma\left(\left(K+1\right)/2\right)}\right)^2 - 1\right) \quad , \tag{20}$$

which is always larger than the CRLB. Next, we can compare both estimators of σ, described above, in terms of the Mean Squared Error (MSE), which is defined as [21]:

$$E\left[(\sigma - \hat{\sigma})^2\right] = \left\{E\left[\hat{\sigma}\right] - \sigma\right\}^2 + \mathrm{Var}(\hat{\sigma}) \quad , \tag{21}$$

or explicitly for the conventional and the ML estimators:

$$\begin{aligned} \mathrm{MSE}_{\hat{\sigma}_{ML}} &\simeq \frac{\sigma^2}{N}\left(\frac{1}{2K} + \frac{1}{16NK^2}\right) & (22)\\ \mathrm{MSE}_{\hat{\sigma}_c} &= \frac{\sigma^2}{N}\left(\frac{K}{2}\left(\frac{\Gamma\left(K/2\right)}{\Gamma\left(\left(K+1\right)/2\right)}\right)^2 - 1\right) & (23)\end{aligned}$$

To compare the conventional estimator with the ML estimator, an MSE ratio is defined as:

$$\mathrm{MSE}_{\mathrm{ratio}} = \frac{\mathrm{MSE}_{\hat{\sigma}_c} - \mathrm{MSE}_{\hat{\sigma}_{ML}}}{\mathrm{MSE}_{\hat{\sigma}_{ML}}} \quad . \tag{24}$$

Note that the MSE ratio is independent of the noise variance. The MSE ratio, as a function of the number of data points, is shown in Fig. 1 for $K = 2, 4$, and 6. For large N, the MSE of the common estimator (19) is significantly larger than that of the ML estimator (13). The performance of the conventional estimator compared to the ML estimator is worst for conventional magnitude MR images, where $K = 2$.

3 Estimation of signal parameters

We now concentrate on the estimation of signal parameters from conventional magnitude MR data. It will be assumed that the model of the signal is known. As an illustrative example we will discuss the estimation of T_1 and T_2 relaxation parameters.

3.1 General considerations

As was discussed above, magnitude data are known to be Rice distributed:

$$p_M(M|f(\theta)) = \frac{M}{\sigma^2} \exp\left(-\frac{M^2 + f^2(\theta)}{2\sigma^2}\right) I_0\left(\frac{f(\theta)M}{\sigma^2}\right) \quad . \tag{25}$$

M denotes the pixel value of the magnitude image. Here, θ represents the parameter vector to be estimated of which the components are generally given by the pseudo proton density ρ, the spin-lattice or longitudinal relaxation constant T_1 and the spin-spin or transversal relaxation constant T_2: $\theta \equiv \{\rho, T_1\}$ or $\theta \equiv \{\rho, T_2\}$. $f(\theta)$ is a function of the parameter vector θ, which is completely determined by the MR imaging sequence applied. For example, for measurement of T_1 relaxation times, commonly a snapshot FLASH imaging sequence is applied, where the magnetization relaxation can be described by:

$$f_i(\theta) = \rho \left|1 - 2\exp\left(-\frac{t_i}{T_1}\right)\right| \quad . \tag{26}$$

where $f_i(\theta)$ denotes the deterministic signal component $f(\theta)$ at time t_i. If the transversal magnetization decay is mono-exponential, and conventional spin-echo imaging is performed, the following model is known to be accurate:

$$f_i(\theta) = \rho \exp\left(-\frac{TE_i}{T_2}\right) \quad . \tag{27}$$

The shape of the Rice distribution is strongly dependent on the signal-to-noise ratio (SNR), where the SNR is defined as the ratio $f(\theta)/\sigma$. It is therefore expected that, whenever parameter estimation techniques that were originally developed for Gauss distributed data are applied to magnitude data, systematic errors will be introduced due to the asymmetry of the Rice PDF, especially at low SNR.

3.2 Errors introduced in T_1 and T_2 estimation

For SNR ratios of ∞, the relaxation behavior is the same as that obtained with Gaussian noise. For typical SNR of 20, 30, the expectation of the relaxation behavior is given by Eq. (26) or Eq. (27), as at such SNR values, the value of $f(\theta)$ differs by only 0.12%, 0.05%, respectively. In general however, the expectation value of the magnitude data is given by:

$$E[M] = \sigma\sqrt{\frac{\pi}{2}} e^{-\frac{f(\theta)^2}{4\sigma^2}} \left[\left(1 + \frac{f(\theta)^2}{2\sigma^2}\right) I_0\left(\frac{f(\theta)^2}{4\sigma^2}\right) + \frac{f(\theta)^2}{2\sigma^2} I_1\left(\frac{f(\theta)^2}{4\sigma^2}\right)\right] \quad . \tag{28}$$

The deviation from $f(\theta)$ becomes more pronounced with decreasing SNR. In Fig. 2, the expectation value $E[M]$ for T_1 and T_2 relaxation is shown for various levels of the SNR. The true time constants were 2000 ms and 100 ms for T_1 and T_2, respectively, and 100 for the pseudo proton density ρ.

3.3 Maximum Likelihood estimation

Now, we clarify the ML approach for the estimation of the unknown parameter vector θ from a set of N independent magnitude data points $\{M_i\}$. The proposed technique consists of maximizing for each pixel position the joint probability density function (PDF), also referred to as the likelihood function, of N Rice distributed data points with respect to θ. The likelihood function of N independent magnitude data points is given by:

$$L(\{M_i\}|\theta) = \prod_{i=1}^{N} p_M\left(M_i|\theta\right) \tag{29}$$

$$\stackrel{\text{Eq.(25)}}{=} \frac{1}{\sigma^{2N}} \exp\left(-\sum_{i=1}^{N} \frac{M_i^2 + f_i\left(\theta\right)^2}{2\sigma^2}\right) \prod_{i=1}^{N} M_i I_0\left(\frac{f_i(\theta)M_i}{\sigma^2}\right) \quad . \tag{30}$$

Maximization of L is equivalent to maximizing $\log L$, as log is a monotonic increasing function:

$$\log(L) = -N\log\sigma^2 - \sum_{i=1}^{N} \frac{M_i^2 + f_i(\theta)^2}{2\sigma^2} + \sum_{i=1}^{N} \log I_0\left(\frac{f_i(\theta)M_i}{\sigma^2}\right) + \sum_{i=1}^{N} \log M_i \quad . \tag{31}$$

For maximization of $\log L$, only the terms that are a function of the unknown parameter vector θ are relevant:

$$\log(L) \sim \sum_{i=1}^{N} \left[\log I_0\left(\frac{f_i(\theta)M_i}{\sigma^2}\right) - \frac{f_i(\theta)^2}{2\sigma^2}\right] \quad . \tag{32}$$

Then the ML estimate for the parameter vector θ is the global maximum of $\log(L)$ with respect to θ:

$$\hat{\theta}_{ML} = \arg\left\{\max_{\theta}\left(\log L\right)\right\} \quad . \tag{33}$$

At high SNR, i.e., when the Rice distribution can be well approximated by a Gauss distribution, the likelihood function becomes:

$$L(\{M_i\}|\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \prod_{i=1}^{N} \exp\left(-\frac{\left(M_i - f_i(\theta)\right)^2}{2\sigma^2}\right) \quad . \tag{34}$$

In that case, it is well known that maximization of $\log L$ with respect to the parameter θ is equivalent to minimizing the quadratic distance E given by:

$$E = \sum_{i=1}^{N} \left[M_i - f_i\left(\theta\right)\right]^2 \quad . \tag{35}$$

This is also generally known as least squares (LS) fitting.

3.4 Experiments and Discussion

To show that a bias is introduced in the estimation of signal parameters, whenever Gauss instead of Rice distributed data are assumed, a simulation experiment was set up. Thereby, real valued data were corrupted with Gauss distributed noise. Zero mean imaginary data were also polluted with Gauss distributed noise with the same standard deviation, after which magnitude data were computed. From 16 Rice distributed data points, obtained in this way, θ was estimated, once using the conventional least squares (LS) fitting procedure, and once using the proposed ML estimation technique. Here, θ was equal to (T_1, ρ) or (T_2, ρ), whether data decayed according to Eq. (26) or (27), respectively. The estimation was repeated 10^5 times for each value of the SNR, which is defined as:

$$\mathrm{SNR} = \frac{\langle f(\theta)\rangle}{\sigma} \quad , \tag{36}$$

with $\langle f(\theta)\rangle$ the average signal value:

$$\langle f(\theta)\rangle = \frac{1}{N}\sum_{i=1}^{N} f_i(\theta) \quad , \tag{37}$$

where $f_i(\theta)$ is given by Eq. (26) or (27) in case of T_1- or T_2-estimation, respectively.

Fig. 3 shows the results for the estimation of $\theta \equiv \{\rho, T_1\}$. The true value for the pseudo proton density was $\rho = 100$ and 2000 ms for the T_1 relaxation constant. Each time, the average value was plotted as a function of the SNR. For clarity, the 95% confidence intervals are omitted: the relative error was of the order of 0.1% for both estimators. Fig. 3a and 3b show the results for the estimation of ρ and T_1, respectively. Both figures clearly demonstrate that the proposed ML technique is more accurate compared to conventional LS estimation. In case of high SNR, opposed to the outcomes of the LS estimator, no bias can be observed for the ML estimator. However, at low SNR (SNR < 5) the ML estimator can be seen to become biased, though the bias is still significantly smaller than that obtained by LS estimation.
Similar reasoning yields for simultaneous estimation of T_2 and ρ. Fig. 4a and 4b show the results for the estimation of ρ and T_2, respectively. The true value for the pseudo proton density was $\rho = 100$, and 100 ms for the T_2 relaxation constant. Also in this case ML estimation outperforms LS estimation in terms of accuracy.

The shape of the likelihood function is shown in Fig. 5 for T_1 (a) and T_2 (b) estimation. It was observed that the two-dimensional $\log(L)$ function has only one maximum, corresponding to the ML estimate of ρ and T_2. The general shape of the likelihood function did not change for different values of the true ρ and T_2 parameters, nor for various SNR. As a result, because of the occurrence of only one maximum of the likelihood function, optimization becomes a very simple task: it can be performed using standard optimization techniques without the risk of getting stuck into a local maximum. Each ML estimate was obtained by maximization of the likelihood function using the downhill simplex method of Nelder and Mead in two dimensions [22].
Finally, we remark that in this experiment a mono-exponentially decaying model was fitted to MR magnitude data points so as to illustrate the consequences of not exploiting the proper data PDF. Obviously, the model can be extended by taking into account additional parameters. In that case, a higher dimensional likelihood function needs to be maximized.

4 Conclusions

In the literature, various methods were described for the estimation of the noise standard deviation or the noise variance from magnitude MR data. In this paper, it has been shown that methods based on Maximum Likelihood estimation are superior in terms of the mean squared error.
In addition, a new technique has been proposed for the estimation of model based signal parameters from magnitude MR data, again based on Maximum Likelihood estimation. As an illustrative example, the estimation of T_1 and T_2 relaxation parameters has been discussed. Compared to existing methods, the proposed technique has clearly been demonstrated to yield more accurate results, especially when the data signal-to-noise ratio decreases. Finally, as the likelihood function has been observed to yield only one maximum, the computational requirements for the maximization were low.

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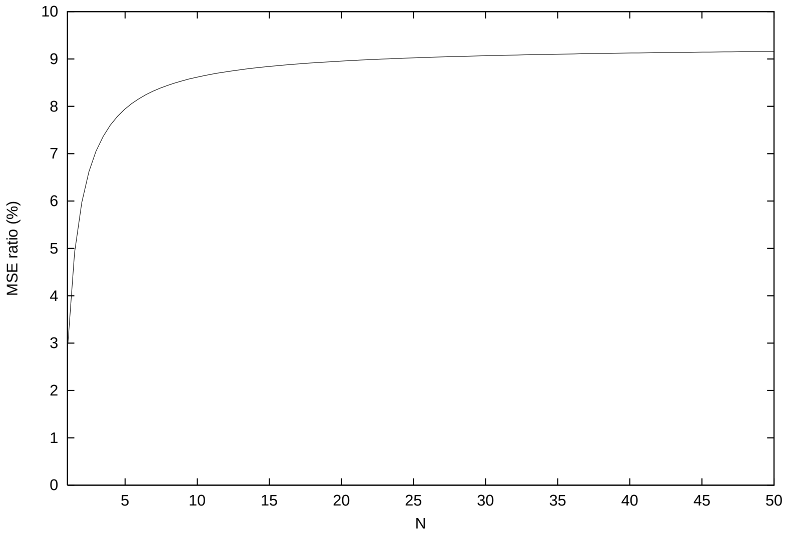
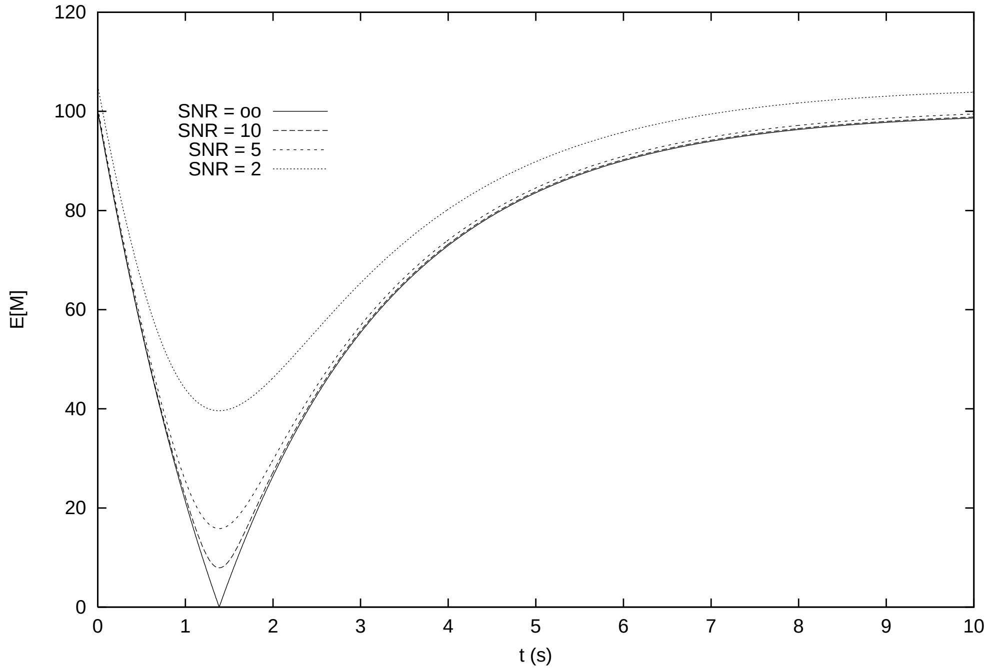
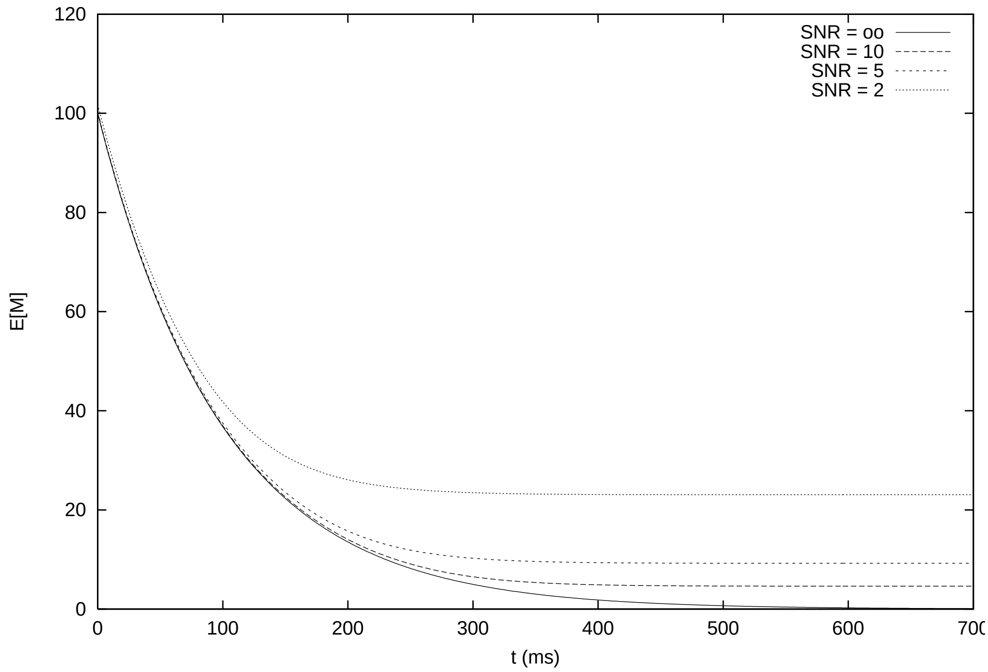


Figure 1: Performance comparison between the conventional and the Maximum Likelihood estimator of the noise standard deviation: Mean Squared Error ratio as a function of the number of data points N, for $K = 2, 4$, and 6.

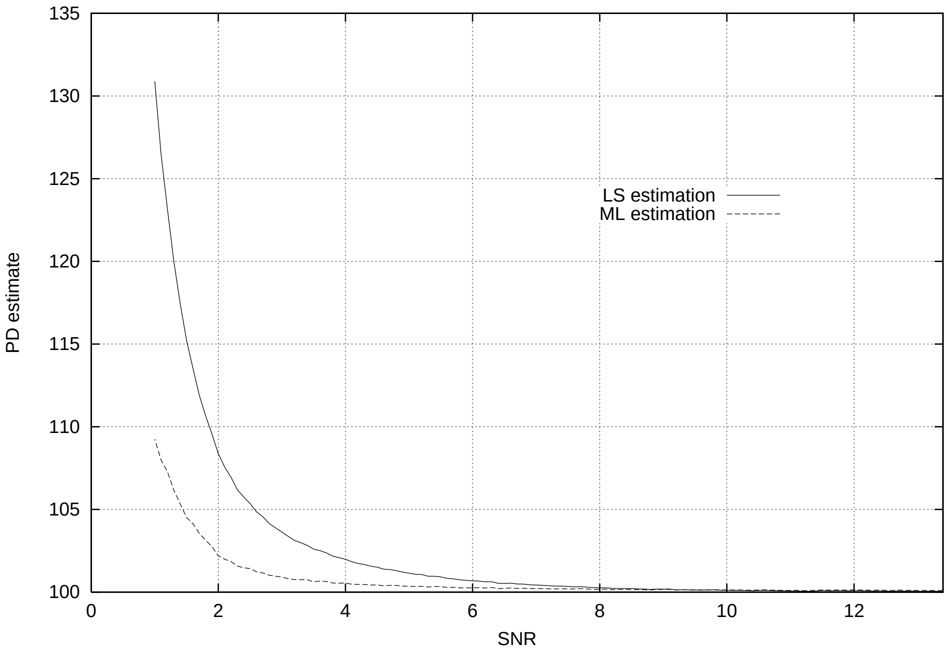


(a) Longitudinal relaxation with $T_1 = 2000$ ms

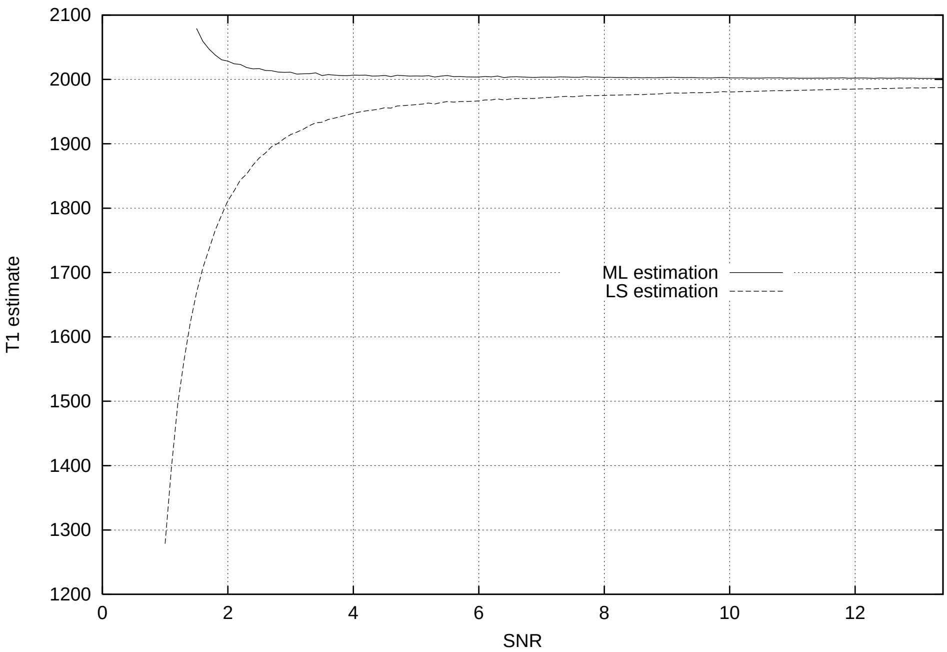


(b) Transversal relaxation with $T_2 = 100$ ms

Figure 2: Expectation values of magnitude MR signal for T_1 and T_2 relaxation as a function of the time t for various values of the SNR.

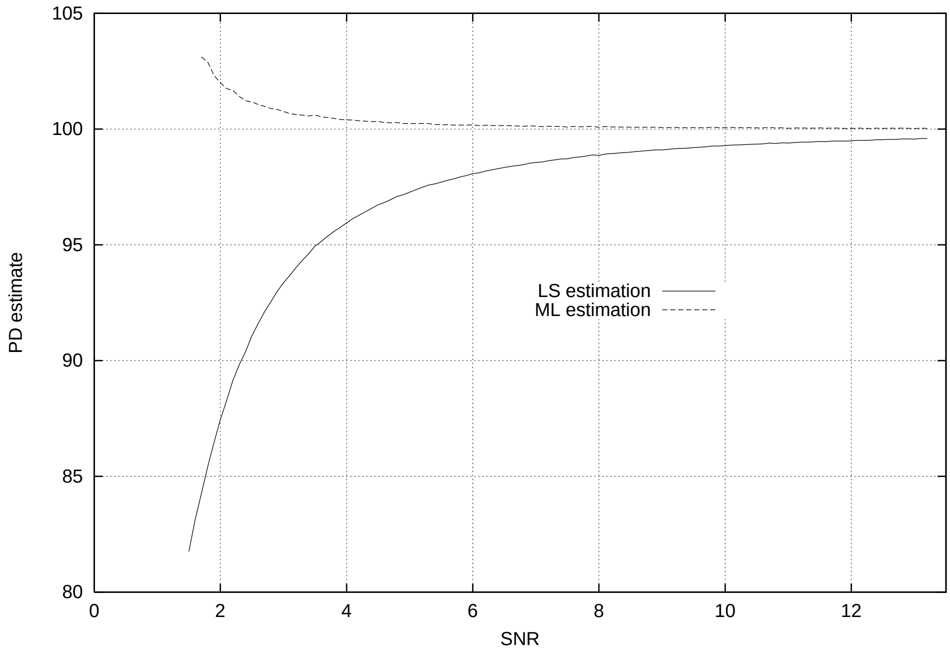


(a) ρ estimation

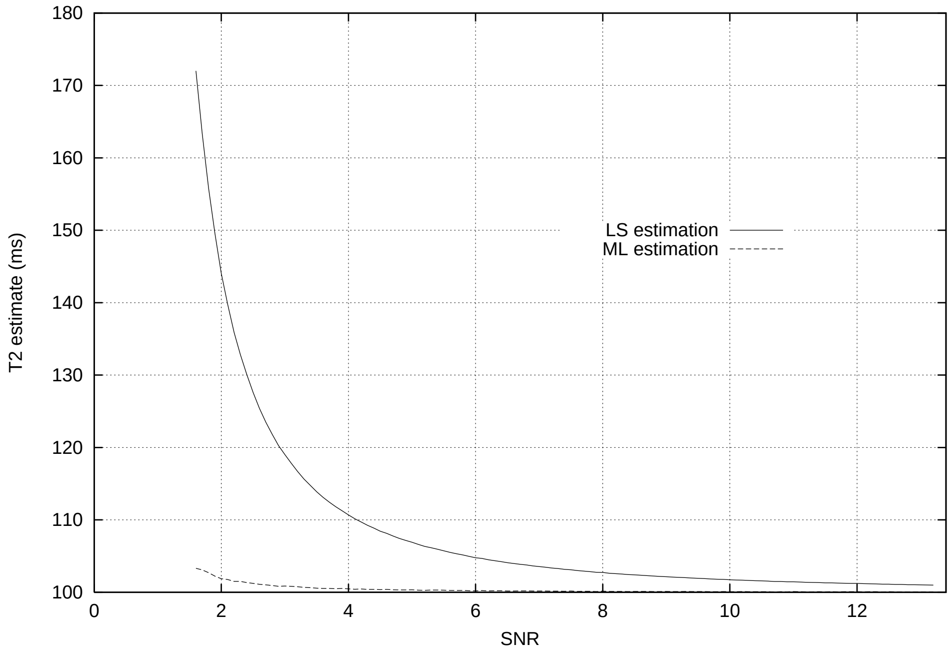


(b) T_1 estimation

Figure 3: Simulation experiment: simultaneous ρ and T_1 estimation as a function of the SNR. The true values are $\rho = 100$ and $\mathrm{T}_1 = 2000$ ms.

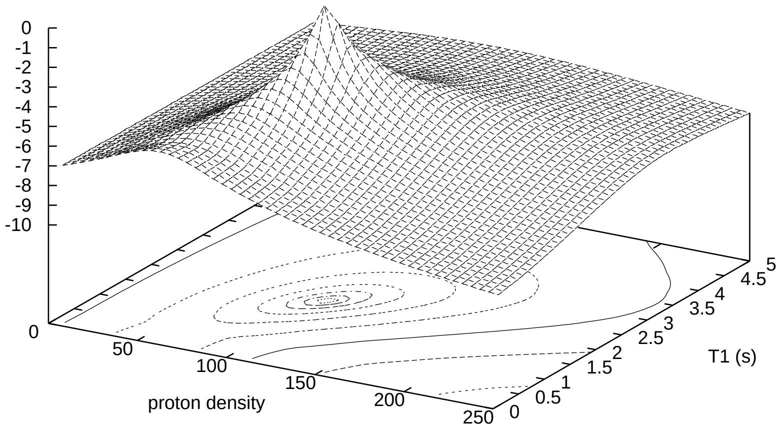


(a) ρ estimation

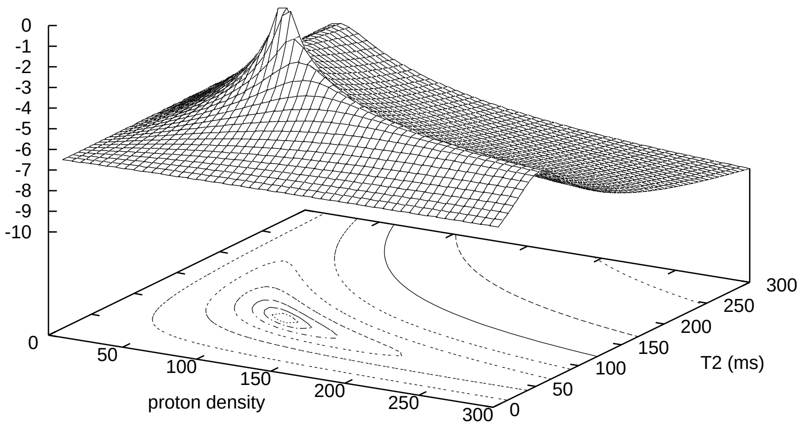


(b) T_2 estimation

Figure 4: Simulation experiment: simultaneous ρ and T_2 estimation from magnitude MR data as a function of the SNR. The true values are $\rho = 100$ and $\mathrm{T}_2 = 100$ ms.



(a) (ρ, T_1) estimation



(b) (ρ, T_2) estimation

Figure 5: Shape of the $\log L$ function as a function of the pseudo proton density and T_1 (a) or T_2 (b).